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ON SOME CLASSES OF LINEAR EQUATIONS, V

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Abstract. The main object of this paper is to establish some conditions which ensure the validity of (1), where A, B are linear operators on a vector space. The obtained results are then applied to the equation $P(L)u = 0$, considered earlier in [**1**], [**2**], [**3**] where P is a polynomial and L a linear operator. This last result is applied to some partial differential equations considered in [**6**].

1. Introduction. The general solution of the differential equation

$$(D-I)(D-2I)u = 0 \qquad (D = d/dx, Iu = u)$$

namely

$$u = C_1e^x + C_2e^{2x} \qquad (C_1, C_2 \text{ arbitrary constants})$$

is the sum of the general solutions of the equations

$$(D-I)u = 0 \quad \text{and} \quad (D-2I)u = 0;$$

in other words

$$\ker(D-I)(D-2I) = \ker(D-I) + \ker(D-2I).$$

Similarly,

$$\ker(D^2-3D+2I)(D-3I) \neq \ker(D^2-3D+2I) + \ker(D-3I)$$

and

$$\ker\frac{\partial^2}{\partial x\partial y} = \ker\frac{\partial}{\partial x} + \ker\frac{\partial}{\partial y},$$

but

$$\ker(D^2-3D+2I)(D-2I) \neq \ker(D^2-3D+2I) + \ker(D-2I)$$

and

$$\ker\frac{\partial^3}{\partial x^2\partial y} \neq \ker\frac{\partial^2}{\partial x\partial y} = \ker\frac{\partial}{\partial x},$$

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where we have supposed that the operators act on sufficiently differentiable functions.

In this paper we shall first examine when will the equality

$$\ker AB = \ker A + \ker B$$

hold where A, B are linear operators on a vector space V, and we shall apply the obtained results to the equation in $u : P(L)u = 0$, where P is a polynomial and L linear operator. This equation was considered in [**1**] – [**3**].

2. The kernel of the product of two linear operators. Suppose that V is a vector space over a field Φ and that $A, B : V \to V$ are linear operators. In this section we shall give three theorems which sure the validity of

$$\ker AB = \ker A + \ker B; \tag{1}$$

the first two give sufficient and the third gives necessary and sufficient conditions for the validity of (1).

Before we prove those theorems notice that:

(i) it is convenient to assume that the operators A and B commute; otherwise a solution of $Au = 0$ need not be a solution of $ABu = 0$,

(ii) if $AB = BA$, then the inclusion

$$\ker A + \ker B \subset \ker AB$$

takes place, and as we know it can be proper;

(iii) if $AB = BA$ and if one of the operators A or B is invertible, than the equality clearly true; we shall (1) therefore always suppose that and A and B are not invertible, i.e. that $\dim\ker A \geq 1$, $\dim\ker A \geq 1$.

THEOREM 1. *If A, B are linear operators mapping V into V such that:*

(i) $AB = BA$;

(ii) $\ker A \cap \ker B = \{0\}$ (0 *is the zero vector of* V)

(iii) $\ker A$, $\ker B$ *and* $\ker AB$ *are finite-dimensional, then*

$$\ker AB = \ker A \oplus \ker B.$$

Proof. From $\ker A \oplus \ker B \subset \ker AB$ follows

$$\dim(\ker A \oplus \ker B) \leq \dim\ker AB.$$

On the other hand for any two operators A and B we have (see, for example, [**4**, p. 135])

$$\dim\ker AB \leq \dim\ker A + \dim\ker B,$$

and since the sum of $\ker A$ and $\ker B$ is direct,

$$\dim\ker A + \dim\ker B = \dim(\ker A \oplus \ker B)$$

implying

$$\dim \ker AB \le \dim(\ker A \oplus \ker B).$$

Hence,

$$\dim \ker AB = \dim(\ker A \oplus \ker B),$$

and since the above subspaces of V are finite-dimensional, this means that

$$\ker AB = \ker A \oplus \ker B,$$

and the proof is complete.

Remark. If the condition (ii) is suppressed, then the equality (1) need not take place. Indeed, the differential equation

$$u''' - 2u'' - 4u' + 8u = 0 \tag{2}$$

with the general solution $u = (C+Dx)e^{2x} + Ee^{-2x}$; C, D, E are arbitrary constants; can be written in the form

$$ABu = 0 \tag{3}$$

where

$$A = \frac{d}{dx} - 2I, \quad B = \frac{d}{dx^2} - 4I \qquad (Iu = u).$$

The conditions (i) and (iii) of Theorem 1 are satisfied, but the condition (ii) is not, and the sum of the general solutions of $Au = 0$ and $Bu = 0$ is not the general solution of (2).

On the other hand, (2) can also be written in the from (3) where

$$A = \frac{d^2}{dx^2} - 4\frac{d}{dx} + 4I, \quad B = \frac{d}{dx} + 2I.$$

In this case all the conditions of Theorem 1 are satisfied and the general solution of (2) is the sum of the general solutions of $Au = 0$ and $Bu = 0$.

Notice, however, that (ii) is not a necessary condition for the validity of (1). Indeed, if V is the space of all real differentiable functions in x and y, $A = \partial/\partial x$, $B = \partial/\partial y$, then $\ker A \cap \ker B$ is the set of all constant functions, but still (1) is true.

Before we formulate the second theorem, we recall that for any linear operator $A : V \to V$ there exists the so-called generalized inverse $\overline{A}$ which is also linear and $A\overline{A}A = A$. See [**5**].

THEOREM 2. *If A, B are linear operators mapping V into V such that*

$$\text{(i)} \quad AB = BA; \qquad \text{(ii)} \quad \overline{A}B = B\overline{A},$$

then the equlity (1) *is valid.*

Proof: The proof of this theorem is based upon the fact that the general solution of the equation $Au = 0$ is $u = t - \overline{A}At$, where $t \in V$ is arbitrary. A proof of this fact is given in [**5**].

Hence, the equations $Au = 0$, $Bu = 0$ have the following general solutions $u = t - \overline{A}At$ and $u = p - \overline{B}Bp$ respectively, where $t, p \in V$ are arbitrary, and their sum is

$$u = t - \overline{A}At + p - \overline{B}Bp \qquad (t, p \in V \text{ arbitrary}). \tag{4}$$

We now prove that the conditions (i) and (ii) imply that a generalized inverse $\overline{AB}$ of AB is $\bar{B}\bar{A}$. Indeed,

$$AB\bar{B}\bar{A}AB = AB\overline{B}\overline{A}BA = AB\overline{B}B\overline{A}A = AB\overline{A}A = BA\overline{A}A = BA = AB,$$

implying that $\overline{AB} = \bar{B}\bar{A}$.

But then the general solution of $ABu = 0$ is

$$u = q - \overline{BA}ABq \qquad (q \in V \text{ arbitrary}) \tag{5}$$

Finally, let us show that the expressions (4) and (5) are equivalent. First, (4) is contained in (5) which is seen by setting $q = t - \overline{A}At + p = \overline{B}Bp$. Conversely, if we put $t = q$, $p = \overline{A}Aq$ in (4) we get

$$u = q - \overline{A}Aq + \overline{A}Aq - \overline{B}B\overline{A}Aq = q - \bar{B}\bar{A}ABq,$$

i.e. (5). Hence, the equality (1) is valid.

Example. Let V be the space of all real functions and define $A : V \to V$ by $Af(x) = f(x) + f(-x)$. A generalized inverse of A is $\overline{A} = I/2$ (I is defined by $If(x) = f(x)$) and hence it commutes with any other linear operator $B : V \to V$. Therefore, if B commutes with A, the general solution of the equation $ABu = 0$ is the sum of the general solutions of the equations $Au = 0$ and $Bu = 0$. For example, let B be the difference operator, i.e.

$$Bf(x) = f(x+1) - f(x).$$

Then the equation $ABf(x) = 0$ becomes

$$f(x+1) - f(x) + f(-x-1) - f(-x) = 0$$

and its general solution is

$$f(x) = P(x) + A(x) - A(-x),$$

where A is an arbitrary function and P is an arbitrary periodic function with period 1.

As a corollary of Theorem 2, we obtain the following result:

If A commutes with its generalized inverse $\overline{A}$, then

$$\ker A^n = \ker A \quad (n \in N).$$

Example. Suppose that M is a square matrix which satisfies the equality $M^2 = M + I$ (I is the unit matrix). The operator A defined by

$$A(X) = MX - XM$$

has a generalized inverse $\overline{A}$ defined by $\overline{A} = (1/5)A$, and hence $A\overline{A} = \overline{A}A$. Therefore the equations $A(X) = 0$ and $A^2(X) = 0$, i.e. $MX - XM = 0$ and

$$2X + MX - 2MXM + XM = 0$$

have the same general solution, namely

$$X = 3T - MT + 2MTM - TM,$$

where T is an arbitrary matrix.

Remark. The condition (ii) of Theorem 2 can be replaced by: $A\overline{B} = \overline{B}A$.

THEOREM 3. *If $A, B : V \to V$ are commutative linear operators, then the conditions*

$$(\forall v \in \ker A)(\exists w \in \ker A)(Bw = v), \quad \textit{i.e.} \quad \ker A \subseteq B(\ker A) \tag{6}$$

and

$$\ker A + \ker B = \ker AB \tag{7}$$

are equivalent.

Proof. The equation

$$ABu = 0 \tag{8}$$

is equivalent to the system

$$Bu = v, \quad Av = 0 \tag{9}$$

According to (6), for any $v \in \ker A$ exists $w = w_v \in \ker A$ such that $Bw_v = v$. Hence, the general solution of the system (9), i.e. of the equation (8) has the form

$$u = w + z, \tag{10}$$

where $w \in \ker A$, and $z \in \ker B$ is arbitrary. Conversely, from (10), where $w \in \ker A$ and $z \in \ker B$ are arbitrary, follows (8), and hence we have proved that (6) implies (7).

In order to prove that (7) implies (6), start with (7) and the negation of (6) which reads

$$(\exists v \in \ker A)(\forall w \in \ker A)(Bw \neq v), \quad \text{i.e.} \quad B(\ker A) \subset \ker A \tag{11}$$

But then (11) implies two possibilities for the system (9):

(i) it has no solutions, which directly contradicts (7); or (ii) the equation $Bu = v$ has a solution $u = w_v \notin \ker A$, and thus the general solution of (9) is $u = w_v + z$ ($z \in \ker B$ arbitrary). But since $w_v \notin \ker A$ and $w_v \notin \ker B$ (this follows from the fact that (11) implies $v \neq 0$) we conclude that $w_v + z \notin \ker A + \ker B$, contradicting (7). Hence we have proved that (7) implies (6).

Example. For an example we again turn to the equation (2) written in the form (3) with

$$A = \frac{d}{dx} - 2I, \qquad B = \frac{d^2}{dx^2} - 4I.$$

The condition (6) is not fulfilled, since $e^{2x} \notin \ker A$, and the solution of $Bu = e^{2x}$ is $u = C_1 e^{2x} + C_e^{-2x} + \frac{1}{4} x e^{2x} \notin \ker A$. On the other hand, if (2) is written in the form (3) with

$$A = \frac{d^2}{dx^2} - 4\frac{d}{dx} + 4I, \quad B = \frac{d}{dx} + 2I,$$

then the condition (6) is satisfied, which is easily verified.

Remark. The condition (6) can be replaced by

$$(\forall v \in \ker B)(\exists w \in \ker B)(Aw = v), \text{ i.e. } \ker B \subset A\,(\ker B)$$

3. Application to the linear equation $P(L)u = 0$. Suppose now that V is a commutative algebra with identity (denoted by i) over R or C, L is a linear operator mapping V into V, and that P is an n-th degree polynomial with coefficients in R or C. In [**1**] we considered the linear equation in u:

$$P(L)u = 0 \tag{121}$$

and obtained its general solution in the form

$$u = \sum_{k=1}^{n} c_k u_k, \tag{13}$$

where $c_k \in \ker L$ are arbitrary, and $Lu_k = \lambda_k u_k$, $P(\lambda_k) = 0$ $(k = 1, \dots, n)$; $\lambda_i \neq \lambda_j$; for $i \neq j$, but we had to make some further assumptions regarding L. Roughly speaking, it was not enough to suppose that L is linear, but something also had to be known about the action of L on the product uv, and we therefore introduced three special classes of operators (e.g. the class $H(V)$ consists of all linear operators $L : V \to V$ which satisfy

$$L(uv) = uLv \Leftrightarrow u \in \ker L.$$

Some important operators (such as the derivatives d/dx, $\partial/\partial x$ or the difference operator Δ) satisfy the above condition, but some simple enough operators, such as d^2/dx^2, $\partial^2/\partial x^2$, $\partial^2/\partial x \partial y$ do not.

We shall now show that (13) is the general solution of (12) with no other supposition on L (except that it is linear), but $c_1, \dots, c_n$ will have a different meaning. We begin with a definition.

Definition. Suppose that u_λ is a characteristic vector of L and that λ is the corresponding characteristic value, i.e. that $Lu_\lambda = \lambda u_\lambda$. We say that $c \in C(\lambda) \subseteq V$ if and only if $L(cu_\lambda) = cLu_\lambda$ $(= \lambda c u_\lambda)$.

The class $C(\lambda)$ is not empty for any λ. Indeed, if a is an arbitrary scalar, then $\alpha i \in C(\lambda)$.

Lemma 1. *Suppose that λ and u_λ hare the same meaning as in the above definition. Then the general solution of the equation*

$$(L - \lambda I)u = 0 \qquad (I \textit{ is the idenlity: } Iu = u) \tag{14}$$

is $u = cu_\lambda$ where $c \in C(\lambda)$ is arbitrary.

Proof. From $Lu = \lambda u$ and $Lu_\lambda = \lambda u_\lambda$ follows

$$\begin{vmatrix} u & u_\lambda \\ Lu & Lu_\lambda \end{vmatrix} = 0,$$

which implies

$$u = cu_\lambda \tag{15}$$

and

$$Lu = cLu_\lambda \tag{16}$$

where $c \in V$. But from (15) follows $Lu = L(cu_\lambda)$ which together with (16) implies $L(cu_\lambda) = cLu_\lambda$. Hence, any solution of (14) has the form (15) with $c \in C(\lambda)$. Conversely, if $c \in C(\lambda)$ it is easily shown that (15) satisfies (14).

Lemma 2. *If $u\mu$ and u_λ are characteristic vectors of the linear operator $L : V \to V$ and if λ and μ are the corresponding characteristic values and $\lambda \neq \mu$, then the general solution of the equation*

$$(L - \lambda I)(L - \mu I)u = 0 \tag{17}$$

is

$$u = c_1 u_\lambda + c_2 u_\mu \tag{18}$$

where $c_1 \in C(\lambda)$ and $c_2 \in C(\mu)$ are arbitrary.

Proof. The operators $A = L - \lambda I$ and $B = L - \mu I$ with $\lambda \neq \mu$ satisfy the conditions of Theorem 3. Indeed, they are clearly commutative. Moreover, for any $c \in C(\lambda)$ the equation $(L - \mu I)u = cu_\lambda$ has a solution, namely $u = \dfrac{c}{\lambda_\mu} u_\lambda$ which belongs to $\ker(L - \lambda I)$, which means that (6) is also fulfilled. Hence, the general solution of (17) is the sum of the general solutions of $(L - \lambda I)u = 0$ and $(L - \mu I)u = 0$, i.e. according to Lemma 1, it is given by (18).

The above proof is easily extended to handle the case of n distinct characteristic values. In other words, we have

Theorem 4. *If $u_{\lambda_1}, \ldots, u_{\lambda_n}$ are characteristic vectors of the lirrear operator $L : V \to V$ and if $\lambda_1, \ldots, \lambda_n$ are the corresponding characteristic values ($\lambda_i \neq \lambda_j$ for $i \neq j$), then the general solution of the equation*

$$\prod_{k=1}^{n} (L - \lambda_k I)u = 0$$

is

$$u = \sum_{k=1}^{n} c_k u_{\lambda_k} \tag{19}$$

where $c_k \in C(\lambda_k)$ are arbitrary $(k = 1, \dots, n)$.

This theorem can be formulated in the following equivalent way:

THEOREM 5. *Suppose that $u_{\lambda_1}, \dots, u_{\lambda_n}$ are characteristic vectors of the linear operator $L : V \to V$ and that $\lambda_1, \dots, \lambda_n$ are the corresponding characteristic values $(\lambda_i \neq \lambda_j$ for $i \neq j)$, which are at the same time zeros of the n-th degree polynomial P. Then the general solution of the equation $P(L)u = 0$ is given by (19), where $c_k \in C(\lambda_k)$ are arbirrary $(k = 1, \dots, n)$.*

4. A result of J. Abramowich. Throughout this section we shall be concerned with functions mapping R^2 into R whose partial derivatives of the required order exist in an open region of R^2. J. Abramowich [**6**] recently showed that the function E_n defined by

$$E_n(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^n} \tag{20}$$

can be used for solving partial differential equations, since for example, $E_2(\lambda xy)$ is the characteristic vector of the operator $D = \partial^2/\partial x \partial y$, i.e. $DE_2(\lambda xy) = \lambda E_2(\lambda xy)$. In connection with that he introduced the following definition:

Definition A. A function $C = C(x, y)$ will be said to be a "D-constant" function if

$$D(CE_2(\lambda xy)) = \lambda CE_2(\lambda xy) \quad (\lambda = \text{const}\,), \tag{21}$$

with $D = \partial^2/\partial x \partial y$, and stated the following theorem[1]:

THEOREM A. *Let $\lambda, \dots \lambda_n$ be distinct roots of $P(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_0 = 0$. Then the general solution of the homogeneous equation $P(D)u = 0$, where $D = \partial^2/\partial x \partial y$, is given by*

$$u(x, y) = \sum_{k=1}^{n} C_k E_2(\lambda_k xy), \tag{22}$$

where C_k are D-constant functions.

There are two comments which we wish to make. First, Definition A is not correct in the sense that the D-constant function defined by (21) clearly depends

[1] Abramowich actually stated his theorem in such a way that it also handles the case of multiple roots of $P(\lambda) = 0$, but we shall not be concerned with that question here.

on λ. In fact, as Abramowich showed himself, the form of a D-constant function is:

$$CE_2(\lambda xy) = g(x) + h(y) + \lambda \int_0^x \int_0^y (g(s) + h(t))E_2(\lambda(x-s)(y-t))dsdt.$$

It would have been better to say that, for example, C is a D_λ-constant function. Also the functions $C_1, \dots, C_n$ which appear in (22) cannot be described simply as D-constant functions, but C_k is a D_{λ_k}-constant function ($k = 1, \dots, n$).

Secondly no proof of Theorem A was given in [**6**] – it was merely stated that the proof is identical to that of the corresponding theorem for ordinary differential equations, as given in [**7**, Chapter VI). However, the proof given in [**7**] is based upon the (previously proved) fact that the general solution of a linear differential equation of order n contains n arbitrary constants. Since no such representation is known for the general solution of the equation $P(D)u = 0$, with $D = \partial^2/\partial x \partial y$, the mentioned proof cannot be used.

Nevertheless, applying Theorem 5, we see that the correct version of Theorem A is as follows:

THEOREM 6. *Let $\lambda_1, \dots, \lambda_n$ be distinct roots of $P(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_0 = 0$. Then the general solution of the homogeneous equation $P(\partial^2/\partial x \partial y)u = 0$ is given by*

$$u(x, y) = \sum_{k=1}^{n} C_k E_2(\lambda_k xy)$$

where $C_1, \dots, C_n$ are arbitrary functions satisfying

$$\frac{\partial^2}{\partial x \partial y} C_k E_2(\lambda_k xy) = \lambda_k C_k E_2(\lambda_k xy) \quad (k = 1, \dots, n).$$

The case when some of the λ_k's are multiple roots of $P(\lambda) = 0$ cannot be treated by this technique.

Remark. It is easily verified that

$$\sum_{k=0}^{\infty} \frac{\lambda^k x^{mk} y^{nk}}{(mk)!(nk)!} \quad (m, n \in N)$$

is a characteristic vector of the operator $\partial^{m+n}/\partial x^m \partial y^n$ and that λ is the corresponding characteristic value. Hence, Theorem 5 can also be applied to linear equations of the form $P(D)u = 0$ where $D = \partial^{m+n}/\partial x^m \partial y^n$.

Remark. Notice that there exist much simpler characteristic vectors u_λ of $\partial^2/\partial x \partial y$ than $E_2(\lambda xy)$ e.g. $u_\lambda = e^{\lambda x + y}$. However, the problem of describing the class of all D_λ-constant functions, i.e. the functions which satisfy

$$\frac{\partial^2}{\partial x \partial y} c u_\lambda = c \frac{\partial^2}{\partial x \partial y} u_\lambda$$

which Abramowich solved in the case where $u_\lambda = E_2(\lambda xy)$ remains open. For the case when $u_\lambda = e^{\lambda x + y}$ it reduces to the equation

$$\frac{\partial^2}{\partial x \partial y} + \frac{\partial c}{\partial x} + \lambda \frac{\partial c}{\partial y} = 0.$$

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