PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 39 (53), 1986, pp. 83–87

NONCOMMUTATIVE VALUATION RINGS

Elbert M. Pirtle

Abstract. Noncommutative valuation rings are duo rings: Every right ideal is a left ideal and conversely. Properties of noncommutative valuation rings are compared to those of commutative valuation rings. Noncommutative valuatiun rings are integrally closed. A noncommutative valuation rings has all the properties of a commutative valuation ring if all its prime ideals are invariant.

In [**8**], Schilling extended the concept of a valuation on a field to that of a division ring as follows.

Definition 1. Let D be a division ring. A valuation on D is a map v from D onto $G \cup \{\infty\}$, where $(G, \cdot, \leq)$ is a totally ordered group and ∞ is an element not in G, which satisfies the following: (i) $v(0) = \infty$; (ii) $v(ab) = v(a)v(b)$; (iii) $v(a + b) \geq \min(v(a), v(b))$ for all $a, b \in D$. (here $g \cdot \infty = \infty = \infty \cdot g$ for all $g \in G$, as in the commutative case)

$V = \{x \in D | \, v(x) \geq e\}$ is a subring of D called the valuation ring of v [**8**]. Lemma 3 of [**8**] shows that valuation rings are duo rings [**1**], [**2**], [**9**]. Neumann [**6**] showed that every totally ordered group (commutative or noncommutative) is the value group of some valuation. Thus the class of noncommutative valuation rings is extensive. The purpose of this paper is to extend the theory of commutative valuation rings to the noncommutative case.

Let R be an (not necessarily commutative) integral domain with identity $1 \neq 0$. R is called a duo ring [**2**] if every right ideal of R is a left ideal and conversely. This is clearly equivalent to the condition that $aR = Ra$ for all $a \in R$. In what follows, R is assumed to be duo. Then R has a left and right division ring D of quotients [**5**]. We let D^* denote the multiplicative group of nonzero elements of D, and U denotes the muliplicative group of units of R. It was shown in [**7**] that U is a normal subgroup of D^*. As in the commutative case, D^*/U is called the group of divisibility of R. D^*/U is partially ordered by defining $xU \leq yU \Leftrightarrow x^{-1}y \in R$, and $(D^*/U, \leq)$ is a directed group [**7**]. (A partially ordered group $(G, \leq)$ is directed

---

AMS Subject Classification (1984): Primary 16A09

if any pair $a, b \in G$ has a lower bound or equivalently an upper bound [**3**].) Clearly $R = \{x \in D|\, 1U \leq xU\}$.

Definition 2. A left (right) R-submodule I of D is called a left (right) fractionary ideal of R if there is $d \in R$, $d \neq 0$, such that $Id \subset R$ $(dI \subseteq R)$.

It follows that if I is a left (right) fractionary ideal of R and $Id \subseteq R$ $(dI \subseteq R)$, then $Id(dI)$ is a right (left) ideal of R and hence an ideal of R. As in the commutative case, the set of principal franctionary ideals of R is $F(R) = \{Rx|\, x \in D, x \neq 0\}$. Since R is duo it follows that $Rx = xR$ for all $Rx \in F(R)$.

We can now state the following theorem [**4**, p. 160].

THEOREM 3. *With R, U, D, D^* as above, the following are equivalent.*

(1) *R is the valuation ring of some valuation v on D.*

(2) *The group of divisibility of R is totally ordered.*

(3) *The set of principal ideals of R is linearly ordered under $\supseteq$*

(4) [**8**] *If $x \in D$, then either $x \in R$ or $x^{-1} \in R$.*

(5) *The set of ideals of R is linearly ordered under $\supseteq$.*

(6) *The set of principal fractional ideals is linearly ordered under $\supseteq$.*

(7) *The set of right fractionary ideals is linearly ordered under $\supseteq$.*

(8) *The set of left fractionary ideals of R is linearly ordered under $\supseteq$.*

Proof. (1)$\Leftrightarrow$(2) is just like the commutative case.

It is easy to see that if U is the multiplicative group of units of R, then $xU \leq yU \Leftrightarrow Rx \supseteq Ry$. Thus $(F(R), \supseteq)$ is order isomorphic to $(D^*/U, \leq)$ under the map $\varphi : D^*/U \to F(R)$ defined by $\varphi(xU) = xR$. Thus (2)$\Leftrightarrow$(6).

Clearly (6)$\Rightarrow$(3).

(3)$\Rightarrow$(6) Let Rx, Ry be principal fractional ideals of R, where $x = ab^{-1}$, $y = cd^{-1}$, $a, b, c, d \in R$. Then $db = bd_0$ for some $d_0 \in R$, and $Rx(bd_0)$ and $Ry(db)$ are principal ideals of R. We may assume that $Rx(bd_0) \subseteq Ry(db)$ and thus $Rx \subseteq Ry$.

(4)$\Leftrightarrow$(2) D^*/U is totally ordered $\Leftrightarrow$ for $x, y \in D^*$, either $xU \leq yU$ or $yU \leq xU \Leftrightarrow$ for $x \in D^*$, either $xU \leq U$ or $U \leq xU \Leftrightarrow x^{-1} \in R$ or $x \in R$ for any $x \in D^*$.

(5)$\Rightarrow$(3) clear.

(3)$\Rightarrow$(5) Let A, B be ideals of R. If $A \subsetneq B$ let $a \in A \setminus B$. For any $b \in B$, $a \notin Rb$, so $b \in Rb \subseteq Ra \subseteq A$, and $B \subseteq A$.

It is clear that (7)$\Rightarrow$(6) and (8)$\Rightarrow$(6) since principal franctionary ideals are both left and right fractionary ideals.

(6)$\Rightarrow$(7) and (6)$\Rightarrow$(8) are similar to (3)$\Rightarrow$(5) and are omitted.

In [**8**] Schilling talks about prime ideals in noncommutative valuation rings but never actually defines them. In [**9**] it was shown that prime ideals in duo rings have the same characterization as prime ideals in commutative rings. We state the following.

Proposition 4. *Let Q be an ideal of R. The folowing are equaivalent.*

(i) *Q is a prime ideal*

(ii) *For $a, b \in R$, if $ab \in Q$, then $a \in Q$ or $b \in Q$*

(iii) *$R \setminus Q$ is a multiplicative system in R.*

As in the commutative case, a multiplicative system is a sybset S of R such that $0 \notin S$ and $s_1 s_2 \in S$ for all $s_1, s_2 \in S$. We have the following corollary to Theorem 1.

Corollary 5. *Let R be a valuation ring and let P be a prime ideal of R. Then R/P is a valuation ring.*

Proof. R/P is duo since R is duo, and R/P is an integral domain since P is prime. The ideals of R/P are linearly ordered since R is a valuation ring.

In [**1**] it was shown that when R is duo then radicals of ideals are characterized exactly as in the commutative case, i.e., if A is an ideal of R, then $\sqrt{A} = \{x \in R | \, x^n \in A\} = \bigcap\{P | \, P \text{ is a prime ideal and } A \subseteq P\}$.

Now, let R be a valuation ring which is not a division ring. Then R is duo and we let D denote the division ring of quotients of R. If G is the value group of the valution v on D which defines R, then $P = \{x \in R | \, v(x) > e\}$ is the unique maximal ideal of R and is prime. With the above notation and assumptions, we have the following [**4**, p. 169].

Theorem 6. *Let A be a proper ideal of R.*

(1) *If A is finitely generated, then A is principal.*

(2) *$\sqrt{A}$ is a prime ideal of R.*

(3) *$\bigcap_{n=1}^{\infty} A^n = P_0$ is a prime ideal of R. If $A^k = A^{k+1}$ for some k, then A is an idempotent prirne ideal.*

(4) *Each prime ideal properly contained in A is contained in P_0.*

(5) *If B is an ideal of R such that $A \subseteq \sqrt{B}$ then B contains a power of A.*

Proof. We only show the first part of (3). The other statements are proved exactly as in the commutative case [**4**, p. 171].

To prove the first part of (3) we show that $R - P_0$ is a multipliactive system in R. So let $x, y \in R\P_0$. Then $x \notin A^n$ and $y \notin A^m$ for some m, n. So $A^n \subset Rx$ and $A^m \subset Ry$. Then $A^n Ry \subset (Rx)(Ry) = Rxy$. So $A^{n+m} \subset Rxy$, and $xy \notin P_0$.

As in the commutative case we say that $x \in D$ is integral over R if there are elements $a_0, a_1, \ldots, a_{n-1}$ such that $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. Since R is duo, the definition is equivalent to $x^n + x^{n-1}b_{n-1} + \cdots + xb_1 + b_0 = 0$ for some $b_0, \ldots, b_{n-1} \in R$.

Definition 7. R is integrally closed if $x \in D$ and x integral over R implies $x \in R$.

Proposition 8. *Valuation rings are integrally closed.*

Proof. Valuation rings are duo rings, and the proof is like the commutative case. Let V be a valuation ring with D as the division ring of quotients of V. Let $x \in D$ be integral over V, say $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_1x + a_0 = 0$. If $x \notin V$ then $x^{-1} \in V$. Then $x^n = -a_0 - a_1x - \cdots - a_{n-1}x^{n-1}$ and $x = -a_0(x^{-1}) - a_1(x^{-1})^{n-2} - \cdots - a_{n-1} \subseteq V$, a contradiction.

Let V be a valuation ring with D as division ring of quotients. Then V is a right and left Bezout domain [**10**], and by the theorem in [**10**], if V' is a ring such that $V \subseteq V' \subseteq D$, then $V' = V_S$ for some saturated multiplicative system S in V. Thus $S = R \setminus \bigcup P_\lambda$ for some collection of prime ideals $\{P_\lambda\}$ of V [**7**]. Since $\{P_\lambda\}$ is totally ordered $S = V - P$ for some prime ideal P of V, and $V' < V_S = V_P$. In [**7**] we constructed quotient rings R_S, where R is an integral domain which is duo and S is a saturated multiplicative system in R. It was shown in [**7**] that R_S is duo if and only if $x^{-1}Sx = S$ for all nonzero $x \in R$ (S is invariant [**7**]). When $S = R - P$, S is invariant if and only if P is invariant. We can now state the following

Propostion 9. *Let V' be a ring such that $V \subseteq V' \subseteq D$. Then* (1) $V' = V_P$ *for some prime ideal P of V.* (2) *V' is a valuation ring $\Leftrightarrow$ P is an invariant prime.*

Schilling in [**8**] showed that there is a 1-1 correspondence between the prime ideals of a valuation ring V and the convex subgroups of the value group G of V. The correspondence is also 1-1 between invariant primes of V and invariant convex subgroups of G.

Let V be a valuation ring with value group G.

Corollary 10. *Every overring of V is a valuation ring $\Leftrightarrow$ every convex subgroup of G is normal.*

Corollary 11. *If G is Abeliean then every overring of V is a valuation ring.*

Corollary 12. *If G satisfies the maximum or minimum condition on normal subgroups then every overring of V is a valuation ring.*

Proof. [**3**, p. 54, Corollary 14].

There exist valuation rings V with prime ideals P that are not invariant.

For let G be a totally ordered group which has a convex subgroup H which is not invariant. (See [**3**, p. 19]). There exists a valuation ring V with G as value group [**6**]. Let P be the prime ideal of V which corresponds to H. Then P is not invariant, and $V_P = \{s^{-1}a| \, s \in V - P, a \in V\} = \{as^{-1}| \, s \in VP, a \in V\}$ is a ring [**7**] with the following properties: (a) V_P is not a duo ring, hence not a valuation ring from [**8**]; (b) $V \subseteq V_P \subseteq D$.

REFERENCES

[1] V. R. Chandran, *On duo-rings*, Lincei-Rend. Sci. Fis. Mat. Natur. **58** (1975), 823–827.

[2] E. H. Feller, *Properties of primary noncommutative rings*, Trans. Amer. Math. Soc. **89** (1958), 79–91.

[3] L. Fuchs, *Partialy Ordered Algebraic Systems*, Addison Wesley, 1963.

[4] R. Gilmer, *Multiplicative Ideal Theory, PartI*, Quin's Papers on Pure Appl. Math. 12, Quieen's Univ., Kingston, Ontario, Canada, 1968.

[5] I. N. Herstein, *Noncommutative Rings*, Carus Mathematical Monographs, 1968.

[6] B. H. Neumann, *On ordered division rings*, Trans. Amer. Math. Soc. **66** (1949), 202–252.

[7] E. M. Pirtle, *Locazation in duo rings*, Publ. Math. Debrecen **31** (1984), 47–52.

[8] O. F. G. Schilling, *Noncommutative valutations*, Bull. Amer. Math. Soc. **51** (1945), 297–304.

[9] G. Thierrin, *On duo rings*, Canad. Math. Bull. **3** (1960), 167–172.

[10] R. A. Beauregard, *Overrings of Bezout domains*, Canad. Math. Bull. **16** (1973).

Department of Mathematics
niversity of Missouri
Kansas City, MI 64110
USA

(Received 12 09 1983)
(Revised 15 07 1985)