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ON QUASIGROUP VARIETIES CLOSED UNDER ISOTOPY

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Abstract. This note deals with quasigroup varieties defined by identities in which every variable occurs exactly twice. We prove that among them only four well-known ones are closed under isotopy.

1. Introduction. Etta Falconer [**3**] has initiated a systematic study of isotopically closed varieties of quasigroups. An ultimate goal, the determination of all such varieties, seems at the moment far from being completed. In this paper we specialize to varieties defined by quadratic identities (identities in which every variable occurs twice) and find all of them which are closed under isotopy.

In group theory the situation with quadratic identities is extremely simple. Namely, every quadratic identity in the language of groups is equivalent (modulo group axioms) to one of the following: $x = x$, $xy = yx$, $x^2 = 1$. Define G_0, A_0, B_0 to be the corresponding group varieties (groups, Abelian groups, Boolean groups). Let G, A, B be varieties consisting of all quasigroups isotopie to a member of G_0, A_0, B_0 respectively. It is well-known that these varieties are axiomatized respectively by quasi-identities $x_1y_1 = x_2y_2 \wedge x_3y_1 = x_4y_2 \wedge x_3y_3 = x_4y_4 \Rightarrow y_1y_3 = x_2y_4$ (the Reidelneister condition), $x_1y_2 = x_2y_1 \wedge x_1y_3 = x_3y_1 \Rightarrow x_2y_3 = x_3y_2$ (the Thomsen condition) and $x_1y_1 = x_2y_{\Rightarrow}x_1y_2 = x_2y_1$; cf. [**1**]. Since every quadratic quasi-identity can be converted into an equivalent (modulo quasigroup axioms) identity, it follows that each of the G, A, B, is defined by a single quadratic identity. Our main result here is that these, together with the variety Q of all quasigroups, are the only isotopically closed varieties defined by quadratic identities:

THEOREM. *Let V be the variety of quasigroups satisfying quadratic identities $E_1, E_2, \ldots$. If V is closed under isotopy, then it is equal to one of Q, G, A, B.*

As a consequence we obtain that every isotopically closed quasigroup variety defined by quadratic identities contains a non-trivial group. Whether the same is true for all isotopically closed varieties is an unsolved problem raised by Falconer [**3**, p. 519].

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Another immediate consequence is that every isotopy invariant quadratic quasigroup identity is equivalent to one of the following: $x = x$, the Reidemeister condition, the Thomsen condition, $x_1y_1 = x_2y_2 \Rightarrow x_1y_2 = x_2y_1$.

2. Preliminaries. Because of some advantages in carrying out inductive arguments, we shall prefer to work with quadratic quasi-identities instead of quadratic identities. The transition from one to the other is immediate and we describe it first.

Let $\Phi = (\varphi_1 = \bigwedge \cdots \bigwedge \varphi_n \Rightarrow \varphi_0)$ be a quadratic quasi-identitety in the quasigroup language and let X be the set of variables occurring in Φ. We say that Φ is connected if for every non-trivial partition $X = X_1 \cup X_2$ there exists a φ_i which contains an occurrence of a variable from both X_1 and X_2. Define aslo Φ to be *reduced* if every φ_i is of the form $x_p \cdot x_q = x_r$ where x_p, x_q, x_r are variables.

Suppose now Φ as above is a connected non-reduced quadratic quasi-identity (in particular, it may be an identity). Then some φ_i is equivalent to $t_1 \cdot t_2 = t_3$, for some quasigroup terms t_1, t_2, t_3 which are not all variables. If $i > 0$, then Φ is equivalent to

$$\Phi' = (\bigwedge_{\substack{j=i \\ j\neq i}} \varphi_j \wedge t_1 = y_1 \wedge t_2 = y_2 \wedge t_3 = y_3 \wedge y_1y_2 = y_3 \Rightarrow \varphi_0)$$

and if $i = 0$ then Φ is equivalent to

$$\Phi' = (\bigwedge_{j=1}^{n} \varphi_j \wedge t_1 = y_1 \wedge t_2 = y_2 \wedge t_3 = y_{\Rightarrow} y_1y_2 = y_3).$$

In both cases Φ' is a connected quadratic quasi-identity. Arguing by induction on the size of terms involved, it easily follows that every connected quadratic quasi-identity is equivalent to a reduced one. From now on we shall use crqq as a shorthand for "connected reduced quadratic quasi-identity".

So let $\Phi = (\varphi_1 \wedge \cdots \wedge \varphi_n \Rightarrow \varphi_0)$ be a crqq and X the set of variables occurring in Φ. We define the graph $\Gamma(Phi)$ associated with Φ as follows. Its vertex set is $\{\varphi_0, \varphi_1, \dots, \varphi_n\} \cup X$ and the edge set is $\{e_{ih} | i \in \{0, \dots, n\}, h \in \{a, b, c\}\}$. The incidence relation is defined by: if $\varphi_i = (x_p \cdot x_q = x_r)$, then e_{ia}, e_{ib}, e_{ic} connect φ_i respectively with x_p, x_q, x_r. Clearly $\Gamma(\Phi)$ is a connected bipartite graph and the degree of every vertex φ_i is three, while every x_p is of degree two.

The following notation is in order to redefine satisfiability of quasi-identities. Let Φ be as above, and for every $\varphi_i = (x_p \cdot x_q = x_r)$ define ${}^\circ\varphi_i = \{x_p, x_q, x_r\}$. A subset Y of X will be called *closed* if for every i the number of edges connecting φ_i with elements of Y is not two. Let $\mathrm{Cl}\, Y$ denote the smallest closed subset of X containing Y. A function $\Theta : Y \to Q$, where Y is a subset of X and Q a quasigroup, will be called a *valuation* (on Y with values in Q) if for every $\varphi_i = (x_p \cdot x_q = x_r)$ such that ${}^\circ\varphi_i \subseteq Y$ one has $\Theta(x_p) \cdot \Theta(x_q) = \Theta(x_r)$. Clearly, every valuation on Y extends uniquely to a valuation on $\mathrm{Cl}\, Y$ if it can be extended there at all.

Let Y contain all but one element of X. One can easily check that a quasigroup Q satisfies Φ iff every valuation $Y \to Q$ extends to a valuation on X cf. [**7**, Lemma 1]. (Notice, as a consequence, that Φ is equivalent to $\Phi_i = (\bigwedge_j \neq i)\varphi_j \Rightarrow \varphi_i)$ for every i. This is in accordance with the obvious fact that $\Gamma(\Phi)$ and $\Gamma(\Phi_i)$ coincide. On the other hand, if the labelling of $\Gamma(\Phi)$ is defined by assigning the label h to every edge e_{ih}, then it is easy to see that the graph $\Gamma(\phi)$ and the labelling of its edges determine $\{\Phi_0, \dots, \Phi_n\}$ uniquely up to renaming of variables). Define a subset Y of X to be a *base* of Φ if the graph obtained from $\Gamma(\Phi)$ by removing for every $y \in Y$ an edge incident with y is a tree. Clearly, if Y is a base, then Y contains no $^\circ\varphi_i$; in fact, bases are maximal subsets of Y with this property. Thus every function on a base with values in a quasigroup is a valuation. On the other hand, one has $\mathrm{Cl}\, Y = X$ for every base Y.

Lemma 1. [**7**, Lemma 2] *Let Y be a base of Φ. A quasigroup Q satisfies Φ if and only if every function $Y \to Q$ extends to a valuation $X \to Q$.*

We remark that graphs associated to various quasigroup equations were introduced and systematically employed in [**7**], to which we refer the reader for more details about the facts stated in this section. The reader will observe a minor difference between $\Gamma(\Phi)$ defined above and that of [**7**] – the former is obtained from the latter by subdividing every edge by a new vertex.

The following lemma provides us with a graphical test for deciding when the variety defined by a crqq consits of group isotopes only; cf. [**1**, Theorem 3], [**4**, Theorem 1] and [**5**, Theorem 4.3].

Lemma 2. [**6**, Theorem 1] *A crqq Φ has the property that every quasigroup satisfying it is a group isotope if and only if the complete graph K_4 on for vertices can be (homeomorphically) embedded in $\Gamma(\Phi)$.*

3. Proof of the Theorem: first part. Assume that V consists of group isotopes only (i.e. $v \subseteq G$). Let L be the variety of all loops contained in V; it can be easily checked that V is the class of all quasigroups isotopic to a member of L; c.f. [**3**, Theorem 3.3]. Since every loop isotopic to a group is a group itself (Albert s theorem), from our assumption $V \subseteq G$ it follows that L is a quadratic group variety. The only such varieties are G_0, A_0 and B_0, so V is one of G, A, B.

It remains to consider the more difficult case $V \nsubseteq G$. Let $\Phi_1, \Phi_2, \dots$ be *crqqs* equivalent to $E_1, E2, \dots$ respectively. In view of Lemma 2 it follows that K_4 cannot be embedded in any $\Gamma(\Phi_i)$.

Notice that every Boolean group satisfies all quadratic identities. Our strategy is to prove that, *given a crgg Φ which is not satisfied by all quasigroups and such that K_4 cannot be embedded in $\Gamma(\Phi)$, there exists an isotope of a Boolean group which does not satisfy Φ.* This would clearly finish the proof of the theorem.

4. Some special isotopes of Boolean groups. If B is a Boolean group and $\alpha, \beta, \gamma \in \mathrm{Aut}\, B$, let $B_{\alpha\beta\gamma}$ be the isotope of B in which the multiplication is defined by $x * y = z$ iff $\alpha(x) + \beta(y) = \gamma(z) = 0$. In this section we determine when a quasigroup of the form $B_{\alpha\beta\gamma}$ satisfies a given crqq.

Let Φ be a crqq and let F be the free group on three free generators a, b, c. Orient edges of $\Gamma(\Phi)$ so that every φ_i is the initial vertex of all edges incident with it. Since every path in $\Gamma(\Phi)$ can be written as a product of oriented edges e_{ih} and their inverses, the labelling function $\lambda(e_{ih}) = h$ can be extended multiplicatively to all paths in $\Gamma(\Phi)$. Thus, for every path ω in $\Gamma(\Phi)$, $\lambda(\omega)$ is a group word in letters a, b, c; we regard it as an element of F. Define $N(\Phi)$ to be the smallest normal subgroup of F which contains labels of all closed paths in $\Gamma(\Phi)$. Let also $f_{\alpha\beta\gamma} : F \to \operatorname{Aut} B$ be the homomorphism defined by $a \mapsto \alpha$, $b \mapsto \beta$, $c \mapsto \gamma$.

LEMMA 3. *The quasigroup $B_{\alpha\beta\gamma}$ satisfies Φ if and only if $N(\Phi) \subseteq \operatorname{Ker} f_{\alpha\beta\gamma}$.*

Proof. Denote the composition $f_{\alpha\beta\gamma} \cdot \lambda$ by f. Call two paths ω_1 and ω_2 conjugate ih there exists a path σ such that $\omega_1 = \sigma\omega_2\sigma^{-1}$. Every closed path in $\Gamma(\Phi)$ is a product of conjugates of simple closed paths (simple = without self-intersections). Therefore $N(\Phi) \subseteq \operatorname{Ker} f_{\alpha\beta\gamma}$ is equivalent to: $f(\omega) = 1$ for every simple closed path ω.

Assume that $B_{\alpha\beta\gamma}$ satisfies Φ and let ω be a simple closed path in $\Gamma(\Phi)$. Let $x_1, \varphi_1, x_2, \varphi_2, \dots, x_k, \varphi_k$ be vertices of ω written in the order one comes across them traversing ω. Since there exists a maximal tree in $\Gamma(\Phi)$ which contains the whole of ω but an edge incident with x_1 (as every tree in a graph is contained in a maximal tree), it follows that there exists a base Y of Φ which contains x_1 and none of $x_2, \dots, x_k$. Define $\Theta_b : Y \to B$ by $\Theta_b(x_1) = b$ and $Theta_b(y) = 0$ for $y \neq x_1$; by Lemma 1 there exists a valuation $\hat{\Theta}_b$ on X which extends Θ_b. Since $\operatorname{Cl}(Y - \{x_1\}) = X - \{x_1, \dots, x_k\}$, it follows that $\hat{\Theta}_b(x) = 0$ for $x \neq x_1, \dots x_k$.

Let e_i and $\acute{e}_i$ be edges of $\Gamma(\Phi)$ connecting φ_i with x_i and x_{i+1}, respectively ($i + 1$ taken mod k). Let e_i'' be the third edge incident with φ_i and z_i be its terminal vertex (Fig. 1), From the fact that $\hat{\Theta}_b$ is a valuation we get $f(e_i)\hat{\Theta}_b(x_i) + f(\acute{e}_i)\hat{\Theta}(x_{i+1}) + f(e_i'')\hat{\Theta}_b(z_i) = 0$. Since $\Theta_b(z_i) = 0$ it follows that $\hat{\Theta}_b(x_i) = f(e_i^{-1}\acute{e}_i)\hat{\Theta}_b(x_{i+1})$. Finally, from these k equalites, $\omega = e_1^{-1}\acute{e}_1 \dots e_k^{-1}\acute{e}_k$ and $\hat{\Theta}_b(x_i) = b$ it follows that $f(\omega)b = b$, i.e., $f(\omega) = 1$, since b was taken arbitrarily. This completes the proof of the "only if" part.

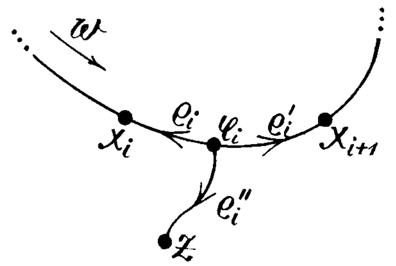

Figure 1.

To prove the converse assume that f vanishes on all closed paths in $\Gamma(\Phi)$. Pick up a base Y of Φ and a function $\Theta : Y \to B_{\alpha\beta\gamma}$. For every $y \in Y$ let $\Theta_y : Y \to B_{\alpha\beta\gamma}$ be defined by $\Theta_y(y) = \Theta(y)$ and $\Theta_y(y') = 0$ for $y' \neq y$. It suffices to prove that every Θ_y extends to a valuation on X. For if $\hat{\Theta}_y$ is a valuation on X extending Θ_y, then $\hat{\Theta} = \sum_{y\in Y} \hat{\Theta}_y$ is a valuation on X extending Θ. Fix a $y \in Y$. We claim that there exists a simple closed path ω with $x_1 = y$ and $x_2, \ldots, x_k \neq Y$, the notation being as in the first part of this proof. (Let T be a maximal tree in $\Gamma(\Phi)$ whose extremal vertices are elements of Y. Let e_1 be that edge incident with y which does not belong to T and ω_1 the shortest path in T connecting the initial and the terminal vertex of e_1. We may take $\omega = e_1^{-1}(\omega_1)$. Define $\hat{\Theta}_y$, by $\hat{\Theta}_y(x) = 0$ for $x \neq x_1, \ldots, x_k$ and $\hat{\Theta}_y(x_i) = f(e_{i-1}^{-1}\acute{e}_{i-1} \ldots e_1^{-1}\acute{e}_1)\Theta_y(x_1)$. Similarly as in the first part of the proof it follows now, using $f(\omega) = 1$, that $\hat{\Theta}_y$, is a valuation on X.

COROLLARY. *If $N(\Phi) \neq \{1\}$, then there exists an isotope of a Boolean group which does not satisfy Φ.*

Proof. In view of Lemma 3, given a non-trivial group word $w(a, b, c)$, we only need to find a Boolean group B and automorphisms α, β, γ of B such that $w(\alpha, \beta, \gamma) \neq 1$. Take B to be the countably infinite Boolean group with independent generators $b_1, b_2, \ldots$. Every permutation of $\{b_i\}$ define an automorphism of B; so $\operatorname{Aut} B$ contains a copy of the countably infinite symmetric group, and hence a copy of the free group F. So there are $\alpha, \beta, \gamma \in \operatorname{Aut} B$ such that, moreover, $B_{\alpha\beta\gamma}$ does not satisfy any Φ with $N(\Phi) \neq \{1\}$.

5. Proof of the Theorem: second part. Let $\Phi = (\varphi_1 \wedge \cdots \wedge \varphi_n)$ be a crqq such that K_4 cannot be embedded in $\Gamma(\Phi)$ and that $N(\Phi) = \{1\}$. In view of the corollary to Lemma 3 and the concluding sentence of Section 3 it suffices to prove, under the assumptions above, that Φ is true on every quasigroup.

First we show that every ${}^\circ\varphi_i$ consists of three elements. Assuming the contrary, there must be two edges e_1, e_2 connecting φ_i with x_p, for some i and p. They bear different labels; so the label of the closed path $e_1 e_2^{-1}$ is $\neq 1$ – a contradiction.

Define a subset A of X to be a *separating set* if A has at most two elements and $\Gamma(\Phi) - A$ has two connected components. From [**7**, Lemma 6] it follows that there exists a separating set whenever $\Gamma(\Phi)$ does not contain an embedded copy of K_4 and has more than two vertices of degree three.

We argue by induction on n. If $n = 1$, then $\Phi = (\varphi_1 \Rightarrow \varphi_0)$ and it is easily checked that coincidence of φ_0 and φ_1 is a necessary and sufficient condition for both $N(\Phi) = \{1\}$ and the satisfiability of Φ by all quasigroups.

So assume $n > 1$, which, as noted above, assures the existence of a separating set. For every separating set A define $\nu(A)$ to be the minimum of the numbers of vertices of degree three in the two components of $\Gamma(\Phi) - A$. Let A_0 be a separating set with the smallest possible ν-value. Let the components of $\Gamma(\Phi) - A_0$ be Γ_1 and Γ_2 with $\nu(A_0) =$ the number of vertices of degree three in Γ_2.

First we prove that A_0 has two elements. Assuming the contrary, let $A_0 = \{x_p\}$, and let φ_i be the vertex of Γ_2 such that $x_p \in^\circ \varphi_i$. Then $^\circ\varphi_i = \{x_p, x_q, x_r\}$ and, since $q \neq r$, it is easy to see that either $B = \{x_q\}$ or $B = \{x_q, x_r\}$ is a separating set and that $\nu(B) < \nu(A_0)$ – a contradiction.

Let then $A_0 = \{x_p, x_q\}$ and φ_i and φ_j be the vertices of Γ_2 such that $x_p \in^\circ \varphi_i$ and $x_q \in^\circ \varphi_j$. We must have $\varphi_i \neq \varphi_j$, because otherwise the third edge x_r of $^\circ\varphi_i$ would make a separating set with $\nu(\{x_r\}) < \nu(A_0)$. Considering x_p and x_q as vertices of both Γ_1 and Γ_2, let $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ be obtained by identifying x_p with x_q in Γ_1 and Γ_2 respectively. Call the new vertices y_1 and y_2. Obviously, both $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are of the form $\Gamma(\Phi')$ for some crqqs Φ' and K_4 cannot be embedded in either of them.

Now $\hat{\Gamma}_2$ contains only two vertices of degree three. For otherwise there would be a separating set in $\hat{\Gamma}_2$. If B is such, and if $B \not\ni y_2$, then B is a separating set for $\Gamma(\Phi)$ too with $\nu(B) < \nu(A_0)$. If $B = \{y_2, x_s\}$, then $B' = \{x_p, x_s\}$ is a separating set for $\Gamma(\Phi)$ with $\nu(B') < \nu(A_0)$.

Thus φ_i and φ_j are the only vertices of degree in Γ_2. Hence $^\circ\varphi_i = \{x_p, x_r, x_s\}$ and $^\circ\varphi_j = \{x_q, x_r, x_s\}$ for some x_r, x_s. We may assume $i, j > 0$; otherwise we would consider an equivalent crqq $\Phi_k = (\bigwedge_{l \neq k} \varphi_l \Rightarrow \varphi_k)$, $k \neq i, j$. The quasi-dentity Φ' obtained by removing φ_i and φ_j from Φ contains one occurrence of each x_p and x_q. Let Φ_1 be obtained from Φ' by replacing x_p and x_q in Φ' by y_1. Then Φ_1 is a crqq written in variables $X - \{x_p, x_q, x_r, x_s\} \cup \{y_1\}$ and $\Gamma(\Phi_1) = \hat{\Gamma}_1$.

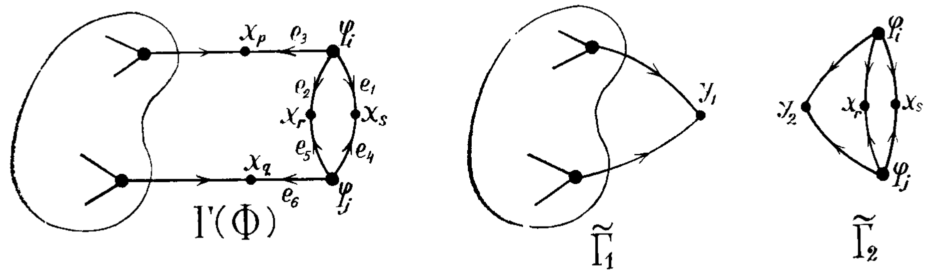

Figure 2.

Let $e_1, \ldots, e_6$ be edges of $\Gamma(\Phi)$ as depicted in Fig. 2. From $\lambda(e_1) \neq \lambda(e_2)$, $\lambda(e_4) \neq \lambda(e_5)$ and $\lambda(e_1 e_4^{-1} e_5 e_2^{-1}) = 1$ it follows that $\lambda(e_1) = \lambda(e_4)$ and $\lambda(e_2) = \lambda(e_5)$, whence also $\lambda(e_3) = \lambda(e_6)$. Therefore, φ_j is obtained from φ_i by replacing x_p by x_q and so $\varphi_i \wedge \varphi_j \Rightarrow x_p = x_q$ is an implication true on all quasigroups. It follows that Φ is a consequence of Φ_1, so Φ is true on all quasigroups provided Φ_1 is.

Thus, it only remains to prove $N(\Phi_1) = \{1\}$. If ω is a closed path in $\hat{\Gamma}_1 = \Gamma(\Phi_1)$ which does not pass through y_1 then ω can be considered as a path in $\Gamma(\Phi)$; so $\gamma(\omega) = 1$. If ω is a simple closed path in $\Gamma(\Phi_1)$ starting at y_1, then either $\omega e_6^{-1} e_4 e_1^{-1}$ or $\omega e_3^{-1} e_1 e_4^{-1} e_6$ is a closed path in $\Gamma(\Phi)$, and since $\gamma(e_6^{1} e_4 e_1^{-1} e_3) = \gamma(e_3^{-1} e_1 e_4^{-1} e_6) = 1$, it follows that $\gamma(\omega) = 1$ in this case too.

Example. Consider the identity $x_1y_1/(x_3 \setminus x_2y_1) = (y_2x_2/x_3) \setminus y_2x_1$. The quasiidentity associated with it is $x_1y_1 = z_1 \wedge x_2y_1 = t_1 \wedge x_3u_1 = t_1 \wedge vu_1 = z_1 \wedge y_2x_1 = z_2 \wedge y_2x_2 = t_2 \wedge u_2x_3 = t_2 \Rightarrow u_2v = z_2$, the graph of which is depicted on Fig. 3. This is the smallest example showing that "$N(\Phi) = \{1\} \Rightarrow \Phi$ is isotopy invariant" is not true in general. (We have just proved this under the additional

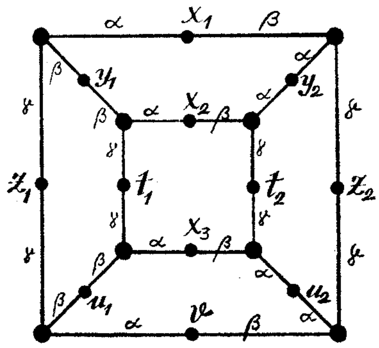

Figure 3.

assumption that K_4 cannot be embedded in $\Gamma(\Phi)$.) Indeed, $N(\Phi) = \{1\}$ is visible from Fig. 3. On the other hand, the identity above is true on every Abelian group, but not on the isotope of the additive group of real numbers defined by $x * y = x + \sqrt[3]{y}$.

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