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ASYMPTOTIC BEHAVIOUR OF SOME COMPLEX SEQUENCES

Slavko Simić

Abstract. We give asymptotic behaviour of a complex sequence with parameters and an application tu Karamata's slowly varying functions.

This article is inspired by the paper [**1**] of D. D. Adamović, in which the following statement has been proved:

A. *For each complex number z and all natural numbers n,*

$$\sum_{k=1}^{n} \frac{z^k}{k^m}\binom{n}{k} \sim \begin{cases} (z+1)^{n+m}/(zn)^m, & \text{if } |z+1|>1 \\ -\log^m n/m!, & \text{if } |z+1| \le 1 \wedge z \neq 0, \end{cases}$$

and $|z+1| \le 1 \wedge z \neq 0$, *more precisely,*

$$\sum_{k=1}^{n} \frac{z^k}{k^m}\binom{n}{k} = -\frac{\log^m n}{m!} - [\log(-z)+C]\frac{\log^{m-1} n}{(m-1)!} + O(\log^{m-2} n), \quad n\to\infty$$

(z complex number, $m \in N$), here C denotes Euleur's constant and the determination of the complex logarithm is such that $\log 1 = 0$.

The first part of that asymptotic estimation was proved by Adamović original method, exposed in [**2**] and [**3**], and the second one using an identity and induction on $n \in N$. Unfortunately, both methods fail if m is a positive real number; so our intention in this article is to establish corresponding – in some cases more precise – results for a generalized sequence, using another method of estimating its integral representation.

1. We shall first study the asymptotic behaviour (as $n \to \infty$) of the complex sequences

$$f_n(z,\alpha,\beta) = \sum_{k=1}^{n} \binom{n}{k} \frac{z^k}{(k+\beta)^\alpha} \quad (n = 1,2,3,\dots)$$

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where α and β are positive reals and z is an arbitrary complex number. Aftewards, we shall show that our results concerning sequences $f_n(z,\alpha,\beta)$ imply and make more precise the statement A.

It appears that the behaviour of $f_n(z,\alpha,\beta)$ fundamentally depends of z being interior, exterior or on the circle $|z+1|=1$. For that reason, we shall formulate our results concerning the asymptotic estimation of $f_n(z,\alpha,\beta)$ in the following propositions:

PROPOSITISON 1. *If* $z+1|>1$, $\alpha>0$, $\beta>0$, *then*

$$f_n(z,\alpha,\beta)\sim(z+1)^{n+\alpha}/(zn)^{\alpha},\ \ n\to\infty.$$

PROPOSITION 2. *If* $|z+1|<1$, $\alpha>0$, $\beta>0$, *then the behaviour of* $f_n(z,\alpha,\beta)$ *is represented by the asymptotic expansion*

$$f_n(z,\alpha,\beta)\sim\frac{1}{\Gamma(\alpha)}\frac{\ln^{\alpha-1}(-zn)}{-zn)^{\beta}}\sum_{k=0}^{\infty}\frac{a_k}{\ln^k(-zn)},\quad n\to\infty,$$

where $a_0=\Gamma(\beta)$, $a_k=(-1)^k\binom{\alpha-1}{k}\Gamma^{(k)}(\beta)$, $(k=1,2,\dots)$ *and* Γ *and* $\Gamma^{(k)}$ *denote the Gamma function and its derivatives.*

PROPOSITION 3. *If* $|z+1|=1\wedge z\neq0$, *then*

(a) *for* $\beta>\alpha>0$ *or* $0<\alpha=\beta<1$: $f_n(z,\alpha,\beta)\sim(z+1)^{n+\alpha}/(zn)^{\alpha}$, $n\to\infty$;

(b) *for* $\alpha>\beta>0$:

$$f_n(z,\alpha,\beta)\sim\frac{1}{\Gamma(\alpha)}\frac{\ln^{\alpha-1}(-zn)}{-zn)^{\beta}}\sum_{k=0}^{\infty}\frac{a_k}{\ln^k(-zn)},\quad n\to\infty;$$

(c) *for* $\alpha=\beta>1$: $f_n(z,\alpha,\beta)\sim\dfrac{1}{\Gamma(\alpha)}\dfrac{\ln^{\alpha-1}(-zn)}{(-zn)^{\beta}}\displaystyle\sum_{k=0}^{[\alpha-1]}\dfrac{a_k}{\ln^k(-zn)}$, $z\to\infty$.

In (b) and (c) the coefficients a_k are defined as before. In all formulae cited above power and logarithm functions have principal values.

Finally we shall give an application of our results to the asymptotic behaviour of sequences of slowly varying functions in Karamata's sense. We remind the reader that if $F_{k-1}(x)=o(F_k(x))$, $S\ni x\to x_0$, $(k=1,2,\dots,n_0)$, and; $G(x)=\sum_{k=1}^{n_0}b_kF_k(x)+o(F_{n_0}(x))$, $S\ni x\to x_0$, then the sum on the right-hand side is called the asymptotic expansion of $G(x)$ as $S\ni x\to x_0$, and we write:

$$G(x)\sim\sum_{k=0}^{n_0}b_kF_k(x),\ \ S\ni x\to x_0;$$

furthermore, if $F_{k-1}(x)=o(F_k(x))$, $S\ni x\to x_0$ $(k=1,2,3,\dots n)$, and

$$G(x)\sim\sum_{k=0}^{n}b_kF_k(x),\ \ S\ni x\to x_0,\ \ (n=1,2,3,\dots),$$

then the series $\sum_{k=1}^{\infty} b_k F_k(x)$ is called the asymptotic expansion of $G(x)$ as $S \ni x \to x_0$, which is denoted by $G(x) \sim \sum_{k=1}^{\infty} b_k F_k(x)$, $S \ni x \to x_0$. We also remark that in the sequel the expression $F(x) = O(G(x))$, $x \in S$, means that there exists a positive constant A (independent of x, but depending probably on some parameters) such that $(\forall x \in S)|F(x)| \leq A|G(x)|$.

In the proofs of Proposition 1–3 we shall use an integral representation of $f_n(z,\alpha,\beta)$, which we obtain in the following way, applying the wellknown integral representation of the Gamma-function:

$$f_n(z,\alpha,\beta) = \sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{(k+\beta)^\alpha} = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \binom{n}{k} z^k \int_0^\infty x^{\alpha-1} e^{-(k+\beta)x} dx =$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\beta x} \sum_{k=0}^{n} \binom{n}{k} (ze^{-x} dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\beta x} (1+ze^{-x})^n dx.$$

So,

$$f_n(z,\alpha,\beta) = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\beta x} (1+ze^{-x})^n dx, \;\; (z \text{ complex } ,\alpha,\beta > 0). \tag{1}$$

Proof of Proposition 1. Let $\xi_n = \ln(1+n^{-1/2})$, $n \in N$. From (1) it follows that:

$$\Gamma(\alpha) f_n(z,\alpha,\beta) = \int_0^\infty x^{\alpha-1} e^{-\beta x} (1+ze^{-x})^n dx =$$

$$\int_0^{\xi_n} x^{\alpha-1} e^{-\beta x} (1+ze^{-x})^n dx + \int_{\xi_n}^\infty x^{\alpha-1} e^{-\beta x} (1+ze^{-x})^n dx = I_1 + I_2.$$

Since $x \mapsto |1+z|e^{-x} + 1 - e^{-x}$ is monotone decreasing for $|z+1| > 1$, we have:

$$I_2 = \int_{\xi_n}^\infty x^{\alpha-1} e^{-\beta x} (1+ze^{-x})^n dx = O\left(\int_{\xi_n}^\infty x^{\alpha-1} e^{-\beta x} |1+ze^{-x}|^n dx \right) =$$

$$= O\left(\int_{\xi_n}^\infty x^{\alpha-1} e^{-\beta x} (|z+1|e^{-x} + 1 - e^{-x})^n dx \right) =$$

$$= O\left(|z+1|e^{-\xi_n} + 1 - e^{-\xi_n})^n \int_{\xi_n}^\infty x^{\alpha-1} e^{\beta x} dx \right) =$$

$$= O\left(\left(|z+1| - \frac{|z+1|-1}{\sqrt{n}+1} \right)^n \right) = O\left(|z+1|^n \exp\left(-\sqrt{n} \frac{|z+1|-1}{|z+1|} \right) \right), n \in N.$$

Since, for $|z+1| \geq 1$, $\ln\left(1+\dfrac{e^x-1}{z+1}\right) = \dfrac{x}{z+1} + O(x^2)$, $x \in (0,\xi_n)$, we get:

$$I_1 = \int_0^{\xi_n} x^{\alpha-1}e^{-\beta x}(1+ze^{-x})^n dx = (z+1)^n \int_0^{\xi_n} x^{\alpha-1}e^{-\beta x}\left(e^{-x}\left(1+\frac{e^x-1}{z+1}\right)\right)^n dx =$$

$$= (z+1)^n \int_0^{\xi_n} x^{\alpha-1}e^{-\beta x}\exp\left(n\ln e^{-x}\left(1+\frac{e^x-1}{z+1}\right)\right)dx =$$

$$= (z+1)^n \int_0^{\xi_n} x^{\alpha-1}e^{-\beta x}\exp\left(n\left(-x+\frac{x}{z+1}+O(x^2)\right)\right)dx =$$

$$= (z+1)^n \int_0^{\xi_n} x^{\alpha-1}e^{-\beta x}\exp\left(-\frac{nzx}{z+1}\right)\exp(O(nx^2))dx =$$

Because, $e^t = 1 + O(te^t)$, $t \in (0,\infty)$, and

$$nx^2 = O(n\xi_n^2) = O(n\ln(1+n^{-1/2})) = O(O(1)) = O(1), \text{ for } x \in (0,\infty),$$

(taking the first O with respect to $x \in (0,\xi_n)$, and the next with respect to $n \in N$), we have further:

$$I_1 = (z+1)^n \int_0^{\xi_n} x^{\alpha-1}e^{-znx/(z+1)}dx + n(z+1)^n \int_0^{\xi_n} x^{\alpha-1}e^{-znx/(z+1)}O(x^2)dx$$

$$e^{O(nx^2)} = (z+1)^n \int_0^{\xi_n} x^{\alpha-1}e^{-znx/(z+1)}dx - (z+1)^n \int_0^{\xi_n} x^{\alpha-1}e^{-znx/(z+1)}dx +$$

$$+(z+1)^n \int_0^{\xi_n} x^{\alpha-1}e^{-znx/(z+1)}O(x^2)dx = I_{12}+I_{13}+I_{14}.$$

Now, because $\operatorname{Re}\dfrac{z}{z+1} = 1 - \dfrac{\cos\arg(z+1)}{|z+1|} > 0$, $|z+1| \geq 1 \wedge z \neq 0$, we get:

$$I_{14} = O\left(n|z+1|^n \int_0^{\xi_n} x^{\alpha+1}e^{-\beta x}\exp\left(-nx\operatorname{Re}\left(\frac{z}{z+1}\right)\right)dx\right) =$$

$$= O\left(n|z+1|^n \int_0^{\xi_n} x^{\alpha+1} exp\left(-\left(\beta+n\operatorname{Re}\frac{z}{z+1}\right)x\right)dx\right) =$$

$$= O\left(\frac{n|z+1|^n}{(\beta+n\operatorname{Re}(z/(z+1))^{\alpha+2}}\right) = O\left(\frac{|z+1|^n}{n^{\alpha+1}}\right), \; n \in N$$

$$I_{13} = O\left(n|z+1|^n \int_{\xi_n}^{\infty} x^{\alpha-1} e^{-\beta x} e^{-nx\mathrm{Re}\,(z/z+1)} dx\right) =$$

$$= O\left(n|z+1|^n e^{-n\xi_n \mathrm{Re}\,(z/z+1)} \int_{\xi_n}^{\infty} x^{\alpha-1} e^{-\beta x} dx\right) =$$

$$= O(|z+1|^n (1+n^{-1/2})^{-n\mathrm{Re}\,z/(z+1)}) = O(|z+1|^n e^{-\sqrt{n}\mathrm{Re}\,(z/z+1)}), \ \ n \in N$$

$$I_{12} = (z+1)^n \int_0^{\infty} x^{\alpha-1} e^{-(\beta+nz/z+1)x} dx = (z+1)^n \frac{\Gamma(\alpha)}{(\beta+nz/(z+1))^{\alpha}}.$$

Finally, we have:

$$I_1 = I_{12} + I_{13} + I_{14} = (z+1)^n \frac{\Gamma(\alpha)}{(\beta+nz/(z+1))^{\alpha}} + O\left(\frac{|z+1|^n}{n^{\alpha+1}}\right),$$
$$\text{for } \ |z+1| \geq 1 \wedge z \neq 0,$$

and hence,

$$f_n(z,\alpha,\beta) = (I_1+I_2)/\Gamma(\alpha) \sim (z+1)^{n+\alpha}/(zn)^{\alpha}, n \to \infty, \ \text{ for } \ |z+1| > 1,$$

which completes the proof.

Proof of Proposition 2. Let $\delta_n = \ln n^{3/4}$, $n \in N$. Then, by (1),

$$\Gamma(\alpha) f_n(z,\alpha,\beta) = \int_0^{\delta_n} + \int_{\delta_n}^{\infty} = J_1 + J_3.$$

Since $x \mapsto |z+1|e^{-x} + 1 - e^{-x}$ is monotone increasing for $|z+1| < 1$, we have:

$$J_1 = \int_0^{\delta_n} x^{\alpha-1} e^{-\beta x} (1ze^{-x})^n dx = O\left(\int_0^{\delta_n} x^{\alpha-1} e^{-\beta x} |(z+1))e^{-x} + 1 - e^{-x}|^n dx\right) =$$

$$= O\left(\int_0^{\delta_n} x^{\alpha-1} e^{-\beta x} (|z+1|e^{-x} + 1 - e^{-x})^n dx\right) =$$

$$= O\left((|z+1|e^{-\delta_n} + 1 - e^{-\delta_n})^n \int_0^{\delta_n} x^{\alpha-1} e^{-\beta x} dx\right) =$$

$$= O(1 - (1-|z+1|)/n^{3/4})^n = O(e^{-n^{1/4}(1-|z+1|)}), \ \ n \in N.$$

Similarly, since for $x \in (\delta_n = \infty)$, $\ln(1+ze^{-x}) = ze^{-x} + O(e^{-2x})$, and

$$e^{nO(\exp(-2x))} = 1 + nO(e^{-2x}) e^{nO(\exp(-2x))} =$$

$$= 1 + O(ne^{-2\delta_n}) e^{O(n\exp(-2\delta_n))} = 1 + O(n^{-1/2}),$$

(taking O's in the first two expressions with respect to $x \in (\delta_n, \infty)$ and in the last two with respect to $n \in N$, we obtain:

$$J_2 = \int\limits_{\delta_n}^{\infty} x^{\alpha-1} e^{\beta x} (1 + z e^{-x})^n dx = \int\limits_{\delta_n}^{\infty} x^{\alpha-1} e^{\beta x} e^{n \ln(1+z \exp(-x))}) dx =$$

$$= \int\limits_{\delta_n}^{\infty} x^{\alpha-1} e^{\beta x} e^{nz \exp(-x)} e^{nO(\exp(-2x))} dx = \int\limits_{\delta_n}^{\infty} x^{\alpha-1} e^{\beta x} e^{nz \exp(-x)} dx +$$

$$+ O(n^{-1/2}) \int\limits_{\delta_n}^{\infty} x^{\alpha-1} e^{\beta x} e^{nz \exp(-x)} dx = I_{23} + I_{24}.$$

Now, putting $n \mathrm{Re}\, z e^{-x} \to x$, we obtain:

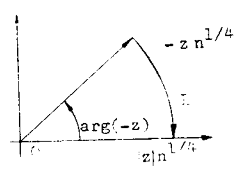
$$J_{24} = O\left(\int\limits_{\delta_n}^{\infty} x^{\alpha-1} e^{-\beta x} e^{n \mathrm{Re}\, z \exp(-x)} dx \right) = O(1/n^{\beta+1/2-\varepsilon}), \;\; n \in N,$$

since we have, with arbitrary chosen $\varepsilon > 0$, $x^{\alpha-1} = O(e^{\varepsilon x})$, $x \in (\delta_n, \infty)$.

Substituting $-zne^{-x} \to w$ in I_{23} (in the sense of forming a complex rectilinear integral, whose calculation leads to J_{23}), and noting that $|z+1| \le 1 \wedge z \ne 0 \Rightarrow \mathrm{Re}\, z < 0$, we get:

$$J_{23} = \frac{1}{(-zn)^{\beta}} \int\limits_{0}^{-zn^{1/4}} e^{-w} \ln^{\alpha-1}\left(\frac{-zn}{w}\right) \cdot w^{\beta-1} dw,$$

the path the of integration being a line segment. Integrating the function $w \mapsto e^{-w} \ln^{\alpha-1} \frac{-zn}{w} w^{\beta-1} dw$ around the contour L (Figure), we get:



$$J_{23} = \frac{1}{(-zn)^{\beta}} \int\limits_{0}^{|z| n^{1/4}} e^{-x} (\ln(-zn) - \ln x)^{\alpha-1} x^{\beta-1} dx + \exp(-O(n^{1/4})), \;\; n \in N,$$

and, as above,

$$J_{23} = \frac{\ln^{\alpha-1}(-zn)}{(-zn)^\beta} \int_0^\infty e^{-x} \left(1 - \frac{\ln x}{\ln(-zn)}\right)^{\alpha-1} x^{\beta-1} dx + \exp) - O(n^{1/4})) =$$

$$= \frac{\ln^{\alpha-1}(-zn)}{(-zn)^\beta} \int_0^\infty x^{\beta-1} e^{-x} \left(\sum_{k=0}^r (-1)^k \binom{\alpha-1}{k} \frac{\ln^k x}{\ln^k(-zn)} + \right.$$

$$\left. + O\left(\frac{\ln^r x}{\ln^r(-zn)}\right)\right) dx + \exp(-O(n^{1/4})) =$$

$$= \frac{\ln^{\alpha-1}(-zn)}{(-zn)^\beta} \left(\sum_{k=0}^r (-1)^k \binom{\alpha-1}{k} \frac{1}{\ln^k(-zn)} \int_0^\infty \ln^k x x^{\beta-1} e^{-x} dx + \right.$$

$$\left. + O\left(\frac{1}{\ln^r(-zn)}\right)\right) + \exp(-O(n^{1/4})) =$$

$$= \frac{\ln^{\alpha-1}(-zn)}{(-zn)^\beta} \left(\sum_{k=0}^r \frac{a_k}{\ln^k(-zn)} + O\left(\frac{1}{\ln^r(-zn)}\right)\right),$$

where,

$$a_0 = \int_0^\infty e^{-x} x^{\beta-1} dx = \Gamma(\beta) \text{ and } a_k = (-1)^k \binom{\alpha-1}{k} \int_0^\infty e^{-x} x^{\beta-1} \ln^k x dx =$$

$$= (-1)^k \binom{\alpha-1}{k} \Gamma^{(k)}(\beta), \quad k = 1, 2, \ldots.$$

So, for $|z+1| < 1$, we have, with the O's taken with respect to $n \in N$,

$$f_n(z,\alpha,\beta) = \frac{1}{\Gamma(\alpha)}(J_1 + J_2) = \frac{\ln^{\alpha-1}(-zn)}{\Gamma(\alpha)(-zn)} \left(\sum_{k=0}^r \frac{a_k}{\ln^k(-zn)} + O\left(\frac{1}{\ln^r(-zn)}\right)\right) +$$

$$+ O\left(\frac{1}{n^{\beta+1/2-\varepsilon}}\right) + \exp(-O(n^{1/4})) \sim \frac{\ln^{\alpha-1}(-zn)}{\Gamma(\alpha)(-zn)^\beta} \sum_{k=0}^\infty \frac{a_k}{\ln^k(-zn)}, \quad n \to \infty.$$

Proof of Proposition 3. For $|z+1| = 1 \wedge z \neq 0$;

$$\Gamma(\alpha) f_n(z,\alpha,\beta) = \int_0^\infty x^{\alpha-1} e^{-\beta x}(1 + ze^{-x})^n dx = \int_0^{\xi_n} + \int_{\xi_n}^{\delta_n} + \int_{\delta_n}^\infty = I_1 + K + J_2,$$

where ξ_n and δ_n are defined as above. Since the previously given estimations for I_1 and J_2 are valid for $|z+1| = 1 \wedge z \neq 0$, we need only estimate the integral K.

Puting $z+1=e^{i\varphi}$, $\varphi\in(o,2\pi)$, we obtain,

$$\begin{aligned} K &= \int_{\xi_n}^{\delta_n} x^{\alpha-1}e^{-\beta x}((z+1)e^{-x}+1-e^{-x})^n dx = \\ &= O\left(\int_{\xi_n}^{\delta_n} x^{\alpha-1}e^{-\beta x}|e^{i\varphi}e^{-x}+1-e^{-x}|^n dx\right) = \\ &= O\left(\int_{\xi_n}^{\delta_n} x^{\alpha-1}e^{-\beta x}\left(1-4\sin^2\frac{\varphi}{2}\cdot e^{-x}(1-e^{-x})\right)^{n/2} dx\right), \ \ n\in N. \end{aligned}$$

As, for $x\in(\xi_n,\delta_n)$, $e^{-x}(1-e^{-x})>e^{-\delta_n}(1-e^{-\delta_n})>1/2n^{3/4}$, we have

$$K=O\left(\int_{\xi_n}^{\delta_n} x^{\alpha-1}e^{-\beta x}\left(1-\frac{2\sin^2\varphi/2}{n^{3/4}}\right)^{n/2}dx\right)=O\left(\exp\left(-\sin^2\frac{\varphi}{2}\cdot n^{1/4}\right)\right).$$

So, under the condition $|z+1|=1\wedge z\neq 1$, for each $r\in N$ fixed,

$$\begin{aligned} &f_n(z,\alpha,\beta)=(I_1+K+J_2)/\Gamma(\alpha)=(z+1)^{n+\alpha}/(zn)^\alpha+ \\ &+\frac{\ln^{\alpha-1}(-zn)}{\Gamma(\alpha)(-zn)^\beta}\left(\sum_{k=0}^{r}\frac{a_k}{\ln^k(-zn)}+O\left(\frac{1}{\ln^{r+1}(-zn)}\right)\right)+ \\ &+O(1/n^{\alpha+1})+O(1/n^{\beta+1/2-\varepsilon}), \ \ n\in N, \end{aligned}$$

which implies the statements of Proposition 3.

Remarks. 1. We note that, $f_n(z,1,1)=\sum_{k=0}^{n}\binom{n}{k}\frac{z^k}{K=1}=\frac{(z+1)^n-1}{(n+1)z}$.

2. One concludes that *Proposition* 1–3 *and the last remark completely determine the asymptotic behaviour of complex sequences* $f_n(z,\alpha,\beta)$.

2. The application of previous results to the sequence: $\sum_1^n\binom{n}{k}\frac{z^k}{k^\alpha}$ $(z\in C,\alpha>0,n\to\infty)$ will give a generalization of Proposition A from [**1**] $(\alpha\in R^+)$.

It is easy to show that $\sum_1^n\binom{n}{k}\frac{z^k}{k^\alpha}=nzf_{n-1}(z,\alpha+1,1)$.

From Proposition 1 it follows that, for $|z+1|>1$, we have:

Proposition 1′ $\displaystyle\sum_{k=1}^{n}\frac{z^k}{k^\alpha}\sim nz\frac{(z+1)^{(n-1)+(\alpha+1)}}{(z(n-1))^{\alpha+1}}\sim\frac{(z+1)^{n+\alpha}}{(zn)^\alpha}$, $n\to\infty$,

Propositions 2 and 3_b imply, for $|z+1|\leq\wedge z\neq 0$ (because $\alpha+>1=\beta$);

Proposition 2′. $\displaystyle\sum_{k=1}^{n}\frac{z^k}{k^\alpha}\sim-\frac{\ln^\alpha(-zn)}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{b_k}{\ln^k(zn)}$, $n\to\infty$

where $b_0 = 1$, $b_k = (-1)^k \Gamma^{(k)}(1)\binom{\alpha}{k}$, $k = 1, 2, \dots$.

Taking the first three terms of the last asymptotic sequence and noting that $\Gamma'(1) = -C$ and $\Gamma''(1) = C^2 + \pi^2/6$, C being Eulers constant, we obtain, for $|z+1| \le 1 \wedge z \ne 0$:

$$\sum_{k=1}^{n} \binom{n}{k} \frac{z^k}{k^\alpha} = -\frac{1}{\Gamma(\alpha+1)} \Big(\ln^\alpha(-zn) - \alpha \Gamma'(1) \ln^{\alpha-1}(-zn) + \\ + \frac{\alpha(\alpha-1)}{2} \Gamma''(1) \ln^{\alpha-2}(-zn) \Big) + O(\ln^{\alpha+3}(-zn)), \ \ n \in N_1$$

that is,

$$\sum_{k=1}^{n} \binom{n}{k} \frac{z^k}{k^\alpha} = -\frac{\ln^{\alpha-1} n}{\Gamma(\alpha+1)} - \frac{\ln^{\alpha-1}}{\Gamma(\alpha)} (\ln(-z) + C) - \frac{\ln^{\alpha-2} n}{\Gamma(\alpha-1)} \Big(\frac{\ln^2(-z)}{2} + \\ + C \ln(-z) + \frac{1}{2} \left(C^2 + \frac{\pi}{6} \right) \Big) + O(\ln^{\alpha-3}(-zn)), \ \ n \in N.$$

This equality includes generalizes and makes more precise the statement in Proposition A. Aljančić shows an interesting application of Proposition 1′ concerning Karamats's slowly varying functions. By definition $L(x) > 0$, $x > 0$, is a slowly varying function (SVF) if $\lim L(tx)/L(x) = 1$, for any $t > 0$. $R(x) > 0$ is called a regularly varying function (RVF) with index α, $\alpha \in R$, if $R(x) = x^\alpha L(x)$. Vuilleumier [**7**] gives necessary and sufficient conditions for an SVF asymptotic relation.

Proposition M′. *If the matrix* (a_{nk}) *satisfies conditions*

$$(1) \ \sum_{n}^{\infty} |a_{nk}| k^\eta = O(n^\eta), \qquad (2) \ \sum_{1}^{n} |a_{nk}| k^{-\eta} = O(n^{-\eta}), \quad (n \to \infty),$$

for some $\eta > 0$, *and if there exist a number* A *such that*

$$(3) \ \sum_{1}^{n} a_{nk} \to A \quad (n \to \infty), \ \ \textit{then} \ L'_n / L_n \to A, \quad (n \to \infty),$$

for any SVF L_n, where $L'_n = \sum_1^\infty a_{nk} L_k$, $(k = 1, 2, \dots)$.

We shall apply our results to Proposition M in the following way:
Let us define a triangular matrix (a_{nk}) by

$$a_{nk} = \begin{cases} \binom{n}{k} \frac{a^k}{k^\alpha} \frac{n^\alpha}{(a+1)^n}, & a > 0, \quad 1 \le k \le n \\ 0, & k > n \end{cases}$$

Proposition 1′ shows that, for the matrix (a_{nk}) defined above, condition (3) is satisfied with $A = ((a+1)/a)^\alpha$. It also shows that

$$n^\eta \sum_{k=0}^{n} |a_{nk}| k^{-\eta} = \sum_{k=0}^{n} a_{nk} \left(\frac{n}{k} \right)^\eta \sim \left(\frac{a+1}{a} \right)^{\alpha+\eta},$$

so condition (2) is valid.

The validity of condition (1) is obvious since the matrix (a_{nk}) is triangular. From Proposition M it follows that:

$$L'_n = \sum_1^\infty a_{nk} L_k \sim A L_n \qquad (n \to \infty), \text{ that is}$$

$$\sum_{k=1}^n \binom{n}{k} \frac{a^k}{k^\alpha} L_k \sim \left(\frac{a+1}{a}\right)^\alpha \frac{(a+1)^n}{n^\alpha} L_n, \quad a, \alpha > 0, \ n \to \infty.$$

Puting $R_n = n^{-\alpha} R_n$, we obtain:

PROPOSITION 4. *For any* RVF R_n *with index*$-\alpha$, $\alpha > 0$*:*

$$\sum_{k=1}^n \binom{n}{k} a^k R_k \sim \left(\frac{a+1}{a}\right) (a+1)^n R_n, \quad a > 0, \quad (n \to \infty)$$

(which, in fact, is true for every |alpha $\in$ *R!).*

The conclusion is that our Proposition 1 is the core of an asymptotic relation for a wide class of sequence.

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Pedagoška akademija,
Ljutice Bogdana 46
11000 Beograd
Jugoslavija

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