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# ON ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATION

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**Abstract.** Necessary and sufficient conditions for oscillation of solutions of the equation

$$y'(t) + \gamma f(t, y(t), y(\Delta_1(t, y(t))), \dots, y(\Delta_n(t, y(t)))) = Q(t), \quad t \geq t_0 \in R, \quad \gamma = \pm 1, \quad n \geq 1$$

are obtained in the case when $Q(t) \equiv 0$ on $[t_0, \infty)$ and sufficient conditions for oscillation and/or nonoscillation are obtained in the case when $Q(t) \not\equiv 0$ on $[t_0, \infty)$. The asymptotic behaviour of oscillatory and nonoscillatory solutions of this equation is studied, too.

In this paper we consider the first order functional differential equation

$$y'(t) + \gamma f(t, y(t), y(\Delta_1(t, y(t))), \dots, y(\Delta_n(t, y(t)))) = Q(t) \tag{1}$$

for $\gamma = \pm 1$, $t \geq t_0 \in R$, which includes as a particular case the equations

$$y'(t) + ay(t - r(y(t))) = 0, \quad a > 0 \tag{2}$$

$$y'(t) - ay(t - h(t, y(t))) = 0, \quad a > 0, \tag{3}$$

used by Cooke [**4**] in modeling infectious diseases and studied in [**4, 5, 14**].

Our main purpose is to obtain necessary and sufficient conditions for oscillation of solutions of (1) when $Q(t) \equiv 0$ for $t \geq t_0$, sufficient conditions for oscillation and/or nonoscillation of all solutions of (1) when $Q(t) \not\equiv 0$ for $t \geq t_0$, and to study the asymptotic behaviour of oscillatory and nonoscillatory solutions of (1) in the cases when $Q(t) \equiv 0$ and $Q(t) \not\equiv 0$ for $t \geq t_0$.

The function $\psi(t) \in C[t_0, \infty)$ is said to be oscillatory if there exists an infinite set $\{\tau_\nu\}_{\nu=1}^{\infty} \subseteq [t_0, \infty)$ of zeros of $\psi(t)$ such that $\tau_\nu \to \infty$ $\nu \to \infty$; otherwise it is said to be nonoscillatory.

An oscillatory function $\psi(t)$ is said to be quickly (moderately) oscillatory if $|\tau_{\nu+1} - \tau_\nu| \to 0$, $\nu \to \infty$ $(\sup\limits_\nu |\tau_{\nu+1} - \tau_\nu| < \infty)$ for any pair of consecutive zeros of $\psi(t)$.

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Further on, we suppose that the functions $f, \Delta_i$ $(i = \overline{1,n})$ and $Q$ are continuous and that the conditions $(H)$ are fulfilled:

H1. $f(t, u_0, u_1, \ldots, u_n) > 0$ $(< O)$ for $u_0 u_i > 0$ $(< 0)$ $(i = \overline{0,n})$ and $t > t_0$.

H2. $\Delta_i(t, v) \to \infty$, for $t \to \infty$, for any fixed $v \in R$, $\Delta_i(t, v) \leq \Delta_i(t, \overline{v})$, for $|v| \leq |\overline{v}|$ $(i = \overline{1,n})$.

We need the following lemmas:

Lemma 1. [**12**]. *Let $\psi(t) \in C^1[t_0, \infty)$ be a quickly oscillatiory function and let $\psi'(t)$ be bounded. Then $\psi(t) \to 0$, for $t \to \infty$.*

Lemma 2. [**13**]. *Let $\psi(t) \in C^1[t_0, \infty)$ be a moderately oscillatory function $\psi'(t) \to 0$, for $t \to \infty$. Then $\psi(t) \to 0$, for $t \to \infty$.*

Lemma 3. *Suppose that the following conditions hold:*

1. *Conditions* (H) *are fulfilled, $Q(t) \equiv 0$ for $t \geq t_0$, $\Delta_i(t, v) \leq t$, for every $v \in R$ $(i = \overline{1,n})$.*
2. *The functions $f(t, ., \ldots, .)$ and $\Delta_i(t, .)$ are Lipshitzian with Lipshitz constants $A > 0$ and $B_i > 0$ $(i = \overline{1,n})$, respectively.*
3. *$f(t, u_0, \ldots, u_n)$ is bounded with respect to every fixed $u_i$ and it is either nondecreasing or nonincreasing in $u_i$ $(i = \overline{1,n})$.*

*Then the necessary and sufficient condition for the existence of a nonoscillatory solution of* (1), *which tends to a nonzero constant as $t \to \infty$, is*

$$\int_{t_0}^{\infty} |(ft, c, \ldots, c)| dt < \infty \qquad \textit{for some } c \neq 0. \tag{4}$$

*Proof.* Necessity. Let $y(t)$ be a nonoscillatory solution of (1) whit $\lim_{t \to \infty} y(t) = a \neq 0$ and let, for instance $a > 0$ (the proof is similar when $a < 0$). Then for each $\varepsilon \in (0, a)$ there exists, $t_1 \geq t_0$ such that $|y(t) - a| < \varepsilon$ for $t \geq t_1$ and by H2 $|y(\Delta_i(t, y(t))) - a| < \varepsilon$ for $t \geq t_2 \geq t_1$ $(i = \overline{1,n})$. Then

$$f(t, y(t), y(\Delta_1(t, y(t))), \ldots, y(\Delta_n(t, y(t)))) \geq f(t, c, \ldots, c) \text{ for } t \geq t_2 \tag{5}$$

where $c = a - \varepsilon$ when $f(t, ., \ldots, .)$ is nondecreasing and $c = a + \varepsilon$ when $f(t, ., \ldots, .)$ is nonincreasing. Integrating (1) from $t_2$ to $t$ and using (5), we get

$$0 = y(t) - y(t_2) + \gamma \int_{t_2}^{t} f(s, y(s), \ldots, y(\Delta_n(s, y(s)))) ds$$

$$\begin{cases} \geq a - \varepsilon - y(t_2) + \displaystyle\int_{t_2}^{t} f(s, c, \ldots, c) ds, & \text{when } \gamma = 1 \\ \leq a + \varepsilon - y(t_2) + \displaystyle\int_{t_2}^{t} f(s, c, \ldots, c) ds, & \text{when } \gamma = -1 \end{cases}$$

which yields (4).

Sufficiency. Let $\gamma = 1$ and (4) hold for $c > 0$ (The proof is similar when $c < 0$). Denote $\delta = c/2$ when $f(t, ., \dots, .)$ nondecreasing and $\delta = c$ when $f(t, ., \cdot, .)$ is nonincreasing. Using (4) and H2 we can find $T_1 \geq t_0$ so that.

$$\int_{T_1}^{\infty} f(t, c, \dots, c)dt \leq \delta \tag{6}$$

and $T_2 = \min_i\{\inf_{t \geq T_1, v \in R} \Delta_i(t, v)\} \geq t_0$. Let $T_0 = \min\{T_1, T_2\}$ and $f_0 = \sup_{t \geq T_0} f(t, c, \dots, c)$.

Denote by $X$ the space of all continuous functions $x : [T_0, \infty) \to R$ with the topology of uniform convergence on compact subintervals $[T_0, \sigma]$ of $[T_0, \infty)$, where $\sigma > T_0$ is an integer, by $Y$ the set of these elements $x \in X$ for which

$$\sigma \geq x(t) \geq 2\delta \text{ for } t \geq T_0 \text{ and } |x(t) - x(\bar{t})| \geq f_0|t - \bar{t}| \text{ for } t, \bar{t} \in [T_0, \infty) \tag{7}$$

and by $\Phi : Y \to X$ the operator, which is defined by the formula

$$(\Phi x)(t) = \begin{cases} 2\delta, & t \in [T_0, T_1] \\ 2\delta - \int\limits_{T_1}^{t} f(s, x(s), x(\Delta_1(s, x(s)))), \dots, x(\Delta_n(s, x(s))))ds, & t \geq T_1. \end{cases}$$

It is easy to see that $X$ is a Frechet space and $Y$ is bounded, convex and closed. Let $x \in Y$. Then $(\Phi x)(t)$ is continuous in $[T_0, \infty)$ and

$$2\delta \geq (\Phi x)(t) \geq 2\delta - \int\limits_{T_1}^{t} f(s, c, \dots, c)ds \geq 2\delta - \int_{T_1}^{\infty} f(s, c, \dots, c)ds \geq \delta \text{ for } t \geq T_0,$$

$|(\Phi x)(t) - (\Phi x)(\bar{t})|$, for $t, \bar{t} \in [T_0, T_1]$

$$|(\Phi x)(t) - (\Phi x)(\bar{t})| = \int\limits_{T_1}^{\bar{t}} f(s, x(s), \dots, x(\Delta_n(s, x(s))))-$$

$$-\int\limits_{T_1}^{t} f(s, x(s), \dots, x(\Delta_n(s, x(s))))ds \geq \int\limits_{T_1}^{\bar{t}} |f(s, x(s), \dots, x(\Delta_n(s, x(s))))|ds \geq$$

$$\geq \int\limits_{t}^{\bar{t}} f(s, c, \dots, c)ds \geq f_0|t - \bar{t}| \text{ for } \bar{t} > t \geq T_1$$

since (6) and (7) hold. Thus $\Phi(Y) \subset Y$ and the functions in $\Phi(Y)$ are equicontinuous on $[T_0, \infty)$ and hence, on compact subintervals $[T_0, \sigma] \subset [T_0, \infty)$.

Let $\{x_\nu\}_{\nu=1}^{\infty} \subset Y$ be uniformly convergent to $x_0$. It is clear that $x_0 \in Y$ and

$|(\Phi x_\nu)(t) - (\Phi x_0)(t)| = 0$ for $t \in [T_0, T_1]$, and

$$(\Phi x_\nu)(t) - (\Phi x_0)(t) \geq \int_{T_1}^{t} |f(s, x_\nu(s), \ldots, x_\nu(\Delta_n(s, x_\nu(s))))-$$

$$-f(s, x_0(s), \ldots, x_0(\Delta_n(s, x_0(s))))|ds \geq \int\limits_{T_1}^{t} F_\nu(s)ds$$

for $t \in [T_1, \sigma]$ when $\sigma > T_1$ and $F_\nu(s) = |f(s, x_\nu(s), \ldots, x_\nu(\Delta_n(s, x_\nu(s)))) - f(s, x_0(s), \ldots, x_0(\Delta_n(s, x_0(s))))|$.

Since $F_\nu(s) \leq 2f(s, c, \ldots, c)$ and

$$F_\nu(s) \leq A\left\{|x_\nu(s) - x_0(s)| + \sum_{i=1}^{n} |x_\nu(\Delta_i(s, x_\nu(s) - x_0(\Delta_i(s, x_0(s)))|\right\} \leq$$

$$A\left\{\|x_\nu(s) - x_0(s)\| + \sum_{i=1}^{n}[|x_\nu(\Delta_i(s, x_\nu(s) - x_\nu\Delta_i(s, x_0(s)))| + |x_\nu(\Delta_i(s, x_0(s)))-\right.$$

$$\left.x_0\Delta_i(s, x_0(s)))|]\right\} \leq A\left\{\|x_\nu(s) - x_0(s)\|_\sigma(n+1) + f_0\sum_{i=1}^{n}|(\Delta_i(s, x_\nu(s)))-\right.$$

$$\left.\Delta_i(s, x_0(s)))|\right\} \leq A\left\{\|x_\nu(s) - x_0(s)\|_\sigma(n+1) + f_0\sum_{i=1}^{n} B_i|x_\nu(s) - x_0(s)|\right\}$$

$$A\left(n + 1 + f_0\sum_{i=1}^{n} B_i\right)\|x_\nu - x_0\|_\sigma \to 0, \quad \nu \to \infty,$$

we conclude according to Lebesgue's dominated convergence theorem, that $\lim\limits_{\nu\to\infty}$ $[\sup_{[T_0,\sigma]} |(\Phi x_\nu)(t) - (\Phi x_0)(t)|] = 0$, i.e. $\Phi$ is a continuous operator.

By Schauder-Tykhonoff fixed point theorem [**6**, p. 9] it follows that there exists $y \in Y$ such that $y = \Phi y$ and the function $y = y(t)$ is a solution of (1) for $t \geq T_1$. Since $y'(t) = -f(s, y(s), \ldots, y(\Delta_n(s, y(s)))) < 0$ for $y \in Y$ and $y(t) \geq \delta$ for $t \geq T_0$, we obtain that there exists $\lim\limits_{t\to\infty} y(t) \neq 0$.

Let $\gamma = -1$. The proof is the same as above, but the operator $\Phi$ is defined by the formula

$$(\Phi x)(t) = \begin{cases} \delta, & t \in [T_0, T_1] \\ \delta + \int\limits_{T_1}^{t} f(s, x(s), x(\Delta_1(s, x(s)))), \ldots, x(\Delta_n(s, x(s))))ds, & t \geq T_1. \end{cases}$$

Lemma 3 is proved.

Theorem 1. *Let conditions of Lemma* 3 *hold. Then the condition*

$$\int\limits_{t_0}^{\infty} |f(t, c, \ldots, c)|dt = \infty, \quad \textit{for any } c \neq 0 \tag{8}$$

*is necessary and sufficient*

1) *either for oscilation or for monotonous convergence to zero as* $t \to \infty$ *of all solutions of* (1) *when* $\gamma = 1$;

2) *for oscilation of all bounded solutions of* (1) *when* $\gamma = -1$.

*Proof.* Necessity. Suppose that (8) is false. Then (4) holds and according to Lemma 3 there exists a nonoscilatory solution of (1) which converges to a nonzero constant, which is a contradiction.

Sufficiency. Let (8) be true for any $c \neq 0$. Suppose that there exists a nonoscillatory solution $y(t)$ of (1) and let, for instance, $y(t) > 0$ for $t \geq t_1 \geq t_0$ when $\gamma = 1$ and $0 < y(t) \leq L$ for $t \geq t_1 \geq t_0$ when $\gamma = -1$ ($L$ =const).

Let $\gamma = 1$. Then H1 and (1) imply that $y'(t) > 0$ for $t \geq t_1$ and there exists $\lim\limits_{t\to\infty} y(t) = k$ for some $k = \text{const} > 0$. If we suppose that $k > 0$ then by Lemma 3 we obtain (4), which is a contradiction.

Let $\gamma = -1$. Then H1 and (1) imply that $y'(t) > 0$ for $t \geq t_1$. Since $y(t)$ is bounded, then $\lim\limits_{t\to\infty} y(t) \neq \text{const} \neq 0$ and by Lemma 3 we obtain (4) which is a contradiction again.

Theorem 1 is thus proved.

THEOREM 2. *Let conditions* ($H$) *hold,* $Q(t) \equiv 0$ *on* $[t_0, \infty)$, $f(t, u_0, \ldots, u_n)$ *be bounded with respect to* $t$ *for every fixed* $u_i$ *and nondecreasing in* $u_i$ ($i = \overline{1,n}$. *Then all bounded quickly oscillatory solutions of* (1) *tend to zero as* $t \to \infty$.

*Proof.* Let $y(t)$ be a bounded quickly oscillatory solution of (1) such that $|y(t)| \leq L$ for $t \geq t_1 \geq t_0$ and $L = \text{const} > 0$. In view of H2 we can find $t_2 \geq t_1$ so that $\Delta_i(t, y(t)) \geq t_1$ for $t \geq t_2$ ($i = \overline{1,n}$) and hence $|y(\Delta_i(t, y(t)))| \leq L$ for $t \geq t_2$. Then

$$f(t, -L, \ldots, -L) \leq f(t, y(t), \ldots, y(\Delta_n(t, y(t))) \leq f(t, L, \ldots, L) \quad \text{for } t \geq t_2$$

and from (1) it follows

$$-f(t, L, \ldots, L) \leq y'(t) \leq -f(t, -L, \ldots, -L) \text{ when } \gamma = 1$$

and

$$-f(t, L, \ldots, L) \leq y'(t) \leq -f(t, L, \ldots, L) \text{ when } \gamma = -1$$

i.e. $y'(t)$ is bounded. By Lemma 1 $y(t) \to 0$, $t \to \infty$, and Theorem 2 is proved.

THEOREM 3. *Let conditions* (H) *hold,* $Q(t) \equiv 0$ *on* $[t_0, \infty)$, $f(t, \cdot, \ldots, \cdot)$ *be nondecreasing and* $\lim\limits_{t\to\infty} |f(t, c, \ldots, c) = 0$ *for any fixed* $c \neq 0$. *Then all bounded moderately oscillatory solutions of* (1) *tend to zero as* $t \to \infty$.

*Proof.* As in the proof of Theorem 2 we find

$$\left.\begin{array}{r} -f(t, L, \ldots, L) \\ f(t, -L, \ldots, -L) \end{array}\right\} \leq y'(t) \leq \left\{\begin{array}{ll} -f(t, -L, \ldots, -L), & \text{when } \gamma = 1 \\ f(t, L, \ldots, L), & \text{when } \gamma = -1 \end{array}\right.$$

and hence $y'(t) \to 0$, $t \to \infty$. By Lemma 2 $y(t) \to 0$, $t \to \infty$, and Theorem 3 is proved.

THEOREM 4. *Let conditions* (H) *and (8) hold,* $Q(t) \equiv 0$ *on* $[t_0, \infty)$ *and* $f(t)$ *be either nondecreasing or nonincreasing. Then* 1) *each nonoscillatory solution of* (1), *for which* $\inf\limits_{t \geq t_0} |y(t)| > 0$, *is unbounded when* $\gamma = 1$*;*

2) *each nonoscillatory solution of* (1) *is unbounded when* $\gamma = -1$.

*Proof.* Suppose the contrary and let $0 < y(t) \leq L$ for $t \geq t_1 \geq t_0$ and $L = \mathrm{const} > 0$. (The proof is similar when $-L \leq y(t) < 0$ for $t \geq t_1 \geq t_0$).

Let $\gamma = 1$. Then there exist $l = \mathrm{const} > 0$ and $t_2 \geq t_1$ such that $y(t \geq l$ for $t \geq t_2$. Via H2 we may find $t_3 \geq t_2$ so that

$$l \leq y(\Delta_i(t, y(t))) \leq L \qquad \text{for } t \geq t_3 \ \ (i = \overline{1,n}). \tag{9}$$

Then (5) holds for $c = l$ when $f(t, \dots)$ is nondecreasing and for $c = L$ when $f(t, \dots)$ is nonincreasing. Integrating (1) from $t_3$ to $t$, using (5) and (9) and letting $t \to \infty$ we obtain the contradiction

$$l \leq y(t) = y(t_3) - \int_{t_3}^{t} f(s, y(s), \dots, y(\Delta_n(s, y(s))))ds \leq y(t_3) -$$

$$- \int_{t_3}^{t} f(s, c, \dots, c)ds \to -\infty, \ \ t \to \infty.$$

Thus $y(t)$ is unbounded.

Let $\gamma = -1$. From (1) via H1 we obtain that $y'(t) > 0$ for $t \geq t_2 \geq t_1$. Since $y(t) > 0$ for $t \geq t_1$ we may find $t_3 \geq t_2$ and $l = \mathrm{const} > 0$ so that $y(t) \geq l$ for $t \geq t_3$. Then as in the proof of the case when $\gamma = 1$ we obtain the contradiction

$$L \geq y(t) = y(t_3) + \int_{t_3}^{t} f(s, y(s), \dots, y(\Delta_n(s, y(s))))ds \geq y(t_3) +$$

$$+ \int_{t_3}^{t} f(s, c, \dots, c)ds \to \infty, \ \ t \to \infty.$$

So, $y(t)$ is unbounded and Theorem 4 is proved.

Now, we shall study the asymptotic behaviour of oscilatory solutions of (1) when $Q(t) \not\equiv 0$ for $t \geq t_0$ and $\gamma = 1$.

LEMMA 4. *Let conditions* (H) *and* (8) *hold,* $f(t, \cdot, \dots, \cdot)$ *be nondecreasing (nonincreasing),* $Q(t) \not\equiv 0$ *for* $t \geq t_0$ *and*

$$\int_{t_0}^{\infty} |Q(t)|dt < \infty. \tag{10}$$

*Then*

$$\lim_{t\to\infty} \inf |y(t)| = 0 \tag{11}$$

*for all (all bounded) solutions of* (1).

*Proof.* Let $f(t,\cdot,\dots,\cdot)$ be nondecreasing and suppose there exists a nonoscilatory solution $y(t)$ of (1) such that $y(t) \geq l$ for $t \geq t_1 \geq t_0$ and some $l = \text{const} > 0$ (The proof is similar when $y(t) \leq -l$ for $t \geq t_1 \geq t_0$). Using H2 we obtain (5) for $t \geq t_2 \geq t_1$ and $c = l$. Integrating (1) from $t_2$ to $t$, using (5) and (10) and taking $t \to \infty$ we obtain the contradiction

$$l \leq y(t) \leq y(t_2) + \int_{t_2}^{t} |Q(s)|ds - \int_{t_2}^{t} f(s,l,\dots,l)ds \to -\infty, t \to \infty.$$

Let $f(t,\cdot,\dots,\cdot)$ be nonincreasing and there exists a bounded nonoscillatory solution $y(t)$ of (1) such that $l \leq y(t) \leq L$ for $t \geq t_1 \geq t_0$ and some $L > l > 0$ (The proof is similar when $l < L < 0$). As above, we obtain (5) for $c = L$ and $t \geq t_2 \geq t_1$. Integrating (1) from $t_2$ to $t$, using (5) and allowing $t \to \infty$ we obtain the contradiction

$$l \leq y(t) \leq y(t_2) + \int_{t_2}^{t} |Q(s)|ds - \int_{t_2}^{t} f(s,L,\dots,L)ds \to -\infty, t \to \infty.$$

Lemma 4 is proved.

THEOREM 5. *If conditions* (H) *and* (10) *hold, then:*

1) *Each oscillatory solution of* (1), *which does not change its sign, tends to zero as* $t \to \infty$.

2) *Each oscillatory solution of* (1), *which changes its sign, tends to zero as* $t \to \infty$ *if the following conditions are fulfilled:*

a) $f(t,\cdot,\dots,\cdot)$ *is nondecreasing and*

$$\int_{t_0}^{\infty} |f(t,c,\dots,c)dt < \infty \quad \textit{for any } c \neq 0; \tag{12}$$

b) $\Delta_i(t,v) \leq t$ $(i = \overline{1,n})$ *for any fixed* $v \in R$;

c) *there exists the uniform on t bound*

$$\varphi(t) = \lim_{|u|\to\infty} \frac{f(t,u,\dots,u)}{u} \quad \textit{such that} \int_{t_0}^{\infty} \tilde{f}(t)dt < \infty$$

$$\textit{where } \tilde{f}(t) \geq \frac{f(t,u,\dots,u)}{u} \textit{ for } u \neq 0.$$

*Proof.* First we will prove that all oscillatory solutions of (1) are bounded. Suppose the contrary, i.e. there exists an unbounded solution $y(t)$ of (1). Then we can find $t_1 \geq t_0$ so that $\Delta_i(t, y(t)) \geq t_0$ $(i = \overline{1,n})$ for $t \geq t_1$ and sets $\{\tau_\mu\}_{\mu=1}^{\infty} \subset [t_1, \infty)$ and $\{\xi_\nu\}_{\nu=1}^{\infty} \subset (\tau_1, \infty)$ of zeros and extremal points of $y(t)$, respectively, with the properties: $\tau_\mu \to \infty$, $\mu \to \infty$; $\xi_\nu \to \infty$, $\nu \to \infty$, and if $M_\nu = |y(\xi_\nu)|$, then $\sup\limits_{[t_0,t_1]} |y(t)| \leq M_1 \leq M_1 \leq \dots$ and $M_\nu \to \infty$, $\nu \to \infty$, (the index $\mu$ may be greater than the index $\nu$, since sticknesses of $y(t)$ with the zero solution are possible).

Let $y(t)$ does not change its sign and let for instance, $y(t) \geq 0$ for $t \geq t_0$ (The proof is similar when $y(t) \leq 0$ for $t \geq t_0$). Via H2 and H1 (1) yields

$$y'(t) \leq Q(t) \qquad \text{for } t \geq t_1. \tag{13}$$

Integrating (13) from $\tau_\mu$ to $\xi_\nu$ and taking $\nu \to \infty$ we get the contradiction

$$\infty > \int_{t_0}^{\infty} |Q(t)| dt \geq M_\nu \to \infty, \quad \nu \to \infty.$$

Let $y(t)$ change its sign and $(\tau_\mu, \tau_{\mu+1}) \ni \xi_\nu$ be its positive semicycle (The proof is similar when $(\tau_\mu, \tau_{\mu+1})$ is a negative semicycle). Let $t_1 \geq t_0$ be chosen so large that $\int_{t_1}^{\infty} \varphi(t) dt < 1/2$. Since $y(t) \leq M_\nu$ and $M_\nu \geq |y(\Delta_i(t, y(t)))|$ $(i = \overline{1,n})$ for $t \in (\tau_\mu, \tau_{\mu+1})$, then

$$f(t, -M_\nu, \dots, -M_\nu) \leq f(t, y(t), \dots, y(\Delta_n(t, y(t)))) \quad \text{for } t \in (\tau_\mu, \tau_{\mu+1}).$$

Integrating (1) from $\tau_\mu$ to $\xi_\nu$, dividing by $M_\nu$ and tending $\nu \to \infty$ we obtain the contradiction

$$\begin{aligned} 1 &\leq \frac{1}{M_\nu} \int_{\tau_\mu}^{\xi_\nu} |Q(t)| dt + \int_{\tau_\mu}^{\xi_\nu} \frac{f(t, -M_\nu, \dots, -M_\nu)}{-M_\nu} dt \leq \\ &\leq \frac{1}{M_\nu} \int_{t_0}^{\infty} |Q(t)| dt + \int_{t_1}^{\infty} \frac{f(t, -M_\nu, \dots, -M_\nu)}{-M_\nu} dt \underset{\nu \to \infty}{\longrightarrow} \int_{t_1}^{\infty} \varphi(t) dt < \frac{1}{2}. \end{aligned}$$

Thus, all oscilatory solutions of (1) are bounded. If we suppose that there exists an oscilatory solution $y(t)$ of (1) such that $\lim\limits_{t \to \infty} \sup |y(t)| = 2m$ for some $m = \text{const} > 0$, then using H2 and (10) we can find numbers $t_0 \leq t_1 \leq \tau_\nu < \xi_\nu$ so, that $\Delta_i(t, y(t)) \geq t_0$ $(i = \overline{1,n})$ for $t \geq t_1$, $\int_{t_1}^{\infty} |Q(T)| dt < m/3\, y(\tau_\nu) = 0$ and $|y(\xi_\nu)| > m$.

Let $y(t)$ does not change its sign on $[t_0, \infty)$. As above we obtain (13). Integrating (13) from $\tau_\nu$ to $\xi_n u$ and having in mind the above assumptions, we obtain the contradiction

$$m \leq \int_{t_1}^{\xi_\nu} |Q(t)| dt \leq \int_{t_1}^{\infty} |Q(T)| dt < \frac{m}{3}.$$

Let $y(t)$ change its sign on $[t_0, \infty)$ and $y(\xi_\nu) > 0$ (The proof is similar when $y(\xi_\nu) < 0$). Let $t_1$ be chosen so large that

$$\int_{t_1}^{\infty} |f(t, -2m, \dots, -2m)|dt < \frac{m}{3}.$$

Integrating (1) from $\tau_\nu$ to $\xi_n u$ and using the assumptions on $f$ and $Q$ we obtain the contradiction

$$m \le \int\limits_{\tau_\nu}^{\xi_\nu} |Q(t)|dt - \int\limits_{\tau_\nu}^{\xi_\nu} f(t, -2m, \dots, -2m)dt \le \int\limits_{t_1}^{\infty} |Q(T)|dt + \\ + \int\limits_{t_1}^{\infty} |f(t, -2m, \dots, -2m)|dt < \frac{2m}{3}.$$

Theorem 5 is thus proved.

THEOREM 6. *Let conditions* (H) *and* (12) *hold, $f(t, \dots)$ be nondecreasing, $Q(t)| > 0$ on $[t_0, \infty)$ and*

$$\int\limits_{t_0}^{\infty} |Q(t)|dt = \infty. \tag{14}$$

*Then all oscillatory solutions of* (1) *are unbounded.*

*Proof.* Let $Q(t) > 0$ on $[t_0, \infty)$ (The proof is similar when $Q(t) < O$ for $t \ge t_0$) and there exists a bounded oscillatory solution $y(t)$ of (1) such that

$$|y(t)| \le c \text{ for } t \ge t_1 \ge t_0 \text{ and } |y(\Delta_i(t, y(t)))| \le c \ (i = \overline{1,n}) \text{ for } t \ge t_2 \ge t_1$$

for some $c > 0$. Then $f(t, y(t), \dots, y(\Delta_n(t, y(t)))) \le f(t, c, \dots, c)$ for $t \ge t_2$ and integrating (1) from $t_2$ to $t$ and using (12) and (14), we obtain the contradiction $c \ge y(t) \ge y(t_2) + \int_{t_2}^{t} Q(s)ds - \int_{t_2}^{t} f(s, c, \dots, c)ds \to \infty, \ t \to \infty$.

Theorem 6 is proved.

Now we will obtain sufficient conditions for nonocillation of all solutions of (1) and we will study their asymptotic behaviour.

THEOREM 7. *Let conditions* (H) *and* (14) *hold, $|Q(t) > 0$ on $[t_0, \infty)$ and conditions* a) -c) *of Theorem* 5 *be fulfilled. Then all solutions of* (1) *are nonoscillatory.*

*Proof.* Let $Q(t) > 0$ on $[t_0, \infty)$ (The proof is similar when $Q(t) < O$ on $[t_0, \infty)$). Suppose there exists an oscillatory solution $y(t)$ of (1). According to Theorem 6, $y(t)$ is unbounded.

Let $t_1$ be a zero of $y(t)$ such that $\Delta_i(t, y(t)) \ge t_0$ $(i = \overline{1,n})$ for $t \ge t_1$ and $\int_{t_1}^{\infty} \varphi(t)dt < \frac{1}{2}$. As in the proof of the first part of Theorem 5 we obtain that $f(t, y(t), \dots, y(\Delta_n(t, y(t)))) \le f(t, M_\nu, \dots, M_\nu)$ for $t \in (\tau_\mu, \tau_{\mu+1})$.

Integrating (1) from $\xi_\nu$ to $\tau_{\mu+1}$, dividing by $M_\nu$ and taking $\nu \to \infty$ we get the contradiction

$$1 \le \int_{\xi_\nu}^{\tau_{\mu+1}} \frac{f(t, M_\nu, \ldots, M_\nu)}{M_\nu} dt \le \int_{t_1}^{\infty} \frac{f(t, M_\nu, \ldots, M_\nu)}{M_\nu} dt \underset{\nu\to\infty}{\longrightarrow} \int_{t_1}^{\infty} \varphi(t)dt < \frac{1}{2}.$$

Theorem 7 is proved.

COROLLARY 1. *Let conditions of Theorem 7 hold. Then all solutions of* (1) *are positive (negative) and unbounded ahore (below) when* $Q(t) > 0$ $(< 0)$ *on* $[t_0, \infty)$.

*Proof.* Let $Q(t) > 0$ on $[t_0, \infty)$ (The proof is similar when $Q(t) < O$ on $[t_0, \infty)$). According to Theorem 7, all solutions of (1) are nonoscillatory. Suppose there exists a solution $y(t) < O$ for $t \ge t_1 \ge t_0$ of (1). From H1 and (1) we obtain $y'(t) \ge Q(t)$ for $t \ge t_1$. Integrating this inequality from $t_1$ to $t$ and taking $t \to \infty$ we obtain the contradiction

$$0 > y(t) \ge y(t_1) + \int_{t_1}^{t} Q(s)ds \underset{\nu\to\infty}{\longrightarrow} y(t_1) + \int_{t_1}^{\infty} Q(s)ds = \infty.$$

Thus all solutions of (1) are positive.

Suppose that $0 < y(t) \le M$ for $t \ge t_2 \ge t_1$ and some $M$ = const $> 0$. Then $y(\Delta_i(t, y(t))) \ge M$ for $t \ge t_2 \ge t_1$ $(i = \overline{1, n})$ and hence $f(t, y(t), \ldots, y(\Delta_n(t, y(t)))) \le f((t, M, \ldots, M)$ for $t \ge t_2$. Integrating (1) from $t_2$ to $t$ using (12) and (14) we obtain the contradiction

$$M \ge \nu(t) \ge y(t_2) + \int_{t_2}^{t} Q(s)ds - \int_{t_2}^{t} f(s, M, \ldots, M)ds \to \infty, \ \ t \le \infty.$$

Corollary 1 is established.

THEOREM 8. *Let conditions* (H) *and* (14), $|Q(t)| > 0$ *for* $t \ge t_0$ *and* $f(t, \cdot, \ldots, \cdot)$ *be nonincreasing. If for any* $c > 0$

$$\int_{t_0}^{\infty} [Q(t) - f(t, c, \ldots, c)] = -\infty \qquad \textit{when } Q(t) > 0 \tag{15}$$

$$\left( \int_{t_0}^{\infty} [Q(t) - f(t, -c, \ldots, -c)]dt = \infty \qquad \textit{when } Q(t) > 0 \right)$$

*then all nonoscillatory solutions of* (1) *are positive (negative) and unbounded above (below).*

*Proof.* As in the proof of the first part of Corollary 1 we establish that the nonoscilatory solutions of (1) are positive. Suppose that $y(t) \leq M$ for $t \geq t_1 \geq t_0$ and $M = \text{const} > 0$. Then $y(\Delta_i(t, y(t))) \leq M$ $(i = \overline{1,n})$ and $f(t, y(t), \ldots, y\Delta_n(t, y(t)))) \geq f(t, M, \ldots, M)$ for $t \geq t_2 \geq t_1$. Integrating (1) from $t_2$ to $t$, tending $t \to \infty$ and using (15) we get

$$0 < y(t) \geq y(t_2) + \int\limits_{t_2}^{t} [Q(s) - f(s, M, \ldots, M)]ds \to -\infty, \;\; t \to \infty.$$

This contradiction proves Theorem 8.

For the equation

$$y'(t) + \gamma f(t, y(\Delta_1(t, y(t)))) = Q(t), \;\; t \geq t_0 \in R, \;\; \gamma = \pm 1, \tag{16}$$

which is a particular case of (1), the following theorem holds:

THEOREM 9. *In addition to* (H2) *for* $n = 1$ *and* (10) *suppose:*

1. $f(t, u) \in C([t_0, \infty) \times R)$, $uf(t, u) > 0$ *for* $u \neq 0$ *and* $t \geq t_0$, $f(t, \cdot)$ *is either nondecreasing when* $\gamma = -1$ *and*

$$0 < \inf_{t > t_0} |f(t, u)| \leq \sup_{t \geq t_0} |f(t, u)| < \infty \quad \textit{for any fixed } u \in R. \tag{17}$$

2. *There exists the derivatives* $\partial\Delta_1(t, v)/\partial t$ *and* $\partial\Delta_1(t, v)/\partial v$ *and they are bounded and nonnegative.*

*Then all nonoscillatory solutions of* (16)*, which are bounded, tend to zero as* $t \to \infty$.

*Proof.* Let $y(t) > 0$ for $t \geq t_1 \geq t_0$ (The proof is similar when $y(t) < 0$ for $t \geq t_1 \geq t_0$). As in the proof of Lemma 2 we establish (11) for all bounded nonoscillatory solutions of (16). Then

$$\lim_{t \to \infty} \inf |y(t, y(t)))| = 0, \quad t \geq t_2 \geq t_1. \tag{18}$$

Suppose

$$\lim_{t \to \infty} \sup |y(\Delta_1(t, y(t)))| > m > 0, \quad t \geq t_2 \geq t_1. \tag{19}$$

In view of (18) and (19), there exists a sequence $\{\lambda_\nu\}_{\nu=1}^{\infty} \subset [t_2, \infty)$ with the following properties: $\lambda_\nu \to \infty$, $\nu \to \infty$ $y(\Delta_1(\lambda_n u, y(\lambda_\nu))) > m$ for all $\nu$ and there exists $\mu_\nu \in (\lambda_\nu, \lambda_{\nu+1})$ such that $y(\Delta_1(\mu_\nu, y(\mu_\nu))) < m/2$ for $\nu \geq 1$.

Let $\alpha_\nu$ be the largest number less than $\lambda_\nu$ such that $m/2 = y(\Delta_1(\alpha_\nu, y(\alpha_\nu)))$ and $\beta_\nu$ be the smallest number greater than $\lambda_\nu$ such that $m/2 = y(\Delta_1(\beta_\nu, y(\beta_\nu)))$ for $\nu \geq 1$. Now in the interval $[\alpha_\nu, \lambda_\nu]$ there exists $\gamma_\nu$ such that

$$\begin{aligned} y'(\Delta_1(\gamma_\nu, y(\gamma_\nu))) \left[ \frac{\partial\Delta_1(\gamma_\nu, y(\gamma_n u))}{\partial t} + \frac{\partial\Delta_1(\gamma_\nu, y(\gamma_n u))}{\partial v} y'(\gamma_\nu) \right] = \\ = \frac{y(\Delta_1(\lambda_\nu, y(\lambda_\nu))) - y(\Delta_1(\alpha_\nu, y(\alpha_\nu)))}{\lambda_\nu - \alpha_\nu} > \frac{m - m/2}{\beta_\nu - \alpha_\nu} = \frac{m}{2(\beta_\nu - \alpha_\nu)} \end{aligned} \tag{20}$$

by the mean value theorem.

But in view of (16), (10) and condition 1 of Theorem 9 we obtain that $y'(t)$, and hence $v'(\Delta_1(t, y(t)))$, are bounded for $t > t_2$. Then via condition 2 of Theorem 9 we obtain the estimate

$$\beta_n u - \alpha_\nu > M \quad \text{for } \nu \geq 1, \ M = \text{ const } > 0. \tag{21}$$

On the other hand, $y(\Delta_1(t, y(t))) \geq m/2$ on $[\alpha_\nu, \beta_\nu]$ because of the way $\alpha_\nu$ and $\beta_\nu$ were chosen. Denote $u = \bigcup_{\nu=1}^{\infty}[\alpha_\nu, \beta_\nu]$. Then

$$f(t, y(\Delta_1(t, y)))) \geq f(t, m/2) \quad \text{for } t \in u \tag{22}$$

when $f(t, \cdot)$ is nondecreasing (The proof is similar when $f(t, \cdot)$ is nonincreasing).

If we suppose that $\int_{t_2}^{\infty} f(t, y(\Delta_1(t, y(t))))dt = \infty$, then from (16) using (10) we obtain the contradiction

$$0 < y(t) \leq y(t_2) + \int_{t_2}^{t} |Q(s)|ds - \int_{t_2}^{t} f(s, y(\Delta_1(s, y(s))))ds \to -\infty, \ t \to \infty.$$

Thus $\int_{t_2}^{\infty} f(s, y(\Delta_1(s, y(s))))ds < \infty$. Using (21) and (22) we get

$$\int_{t_2}^{\infty} f(s, y(\Delta_1(s, y(s))))ds \geq \int_{u} f(s, y(\Delta_1(s, y(s))))ds \geq \int_{u} f(s, m/2)ds =$$

$$\sum_{\nu=1}^{\infty} \int_{\alpha_\nu}^{\beta_\nu} f(s, m/2)ds > \sum_{\nu=1}^{\infty} f_0(\beta_\nu - \alpha_\nu) > f_0 M \lim_{n\to\infty} \sum_{\nu=1}^{n} \nu = \infty$$

where $f_0 = \inf t \geq t_2 f\left(t, \frac{m}{2}\right)$.

This contradiction proves Theorem 9.

*Remark.* Theorem 9 is proved by the techniquc of Chen [**3**].

We note that sufficient conditions for oscillation of sclutions of first order fuctional differential equations have been obtained in [**1, 2, 7 -11, 15**] and in the papers cited in [**7, 15**]. Asymptotic behaviour of oscillatory and nonoscillatory solutions of cited equations is not studied yet.

Finally we shall apply Theorems 1, 2 and 4 to the equations (2) and (3).

Consider equation (2). It is a particular case of (1) with $\gamma = 1$, $f(t, u) = au$ and $\Delta(t, v) = t - r(v)$. If $r(v)$ is Lipschitzian and $r(v) \geq r(\overline{v})$ for $|v| \leq |\overline{v}|$ then all solutions of (2) are either oscillatory or tend monotonously to zero as $t \to \infty$ according to Theorem 1. By Theorem 2 all bounded quickly oscillatory solutions of (2) tend to zero as $t \to \infty$ and by Theorem 4 every nonoscillatory solution $y(t)$ of (2), for which $\inf_{t \geq t_0} |y(t)| > 0$, is unbounded.

Consider equation (3). It is a particular case of (1) with $\gamma = -1$, $f(t,u) = au$ and $\Delta(t,v) = t - h(t,v)$. If $h(t,\cdot)$ is Lipschitzian and $h(t,v) \geq h(t,\overline{v})$ for $|v| \leq |\overline{v}|$ then according to Theorem 1 all bounded solutions of (3) are oscillatory, by Theorem 2 all bounded quickly oscillatory solutions of (3) tend to zero as $t \to \infty$ and by Theorem 4 all nonoscillatory solutions of (3) are unbounded.

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