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A CHARACTERIZATION OF STRICTLY CONVEX METRIC LINEAR SPACES

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Abstract. A subset G of a metric linear space (E, d) is said to be semi-Chebyshev if each element of E has at most approximation in G and the space (E, d) is said to be strictly convex if $d(x, 0) \leq r$, $d(y, 0) \leq r$ imply $d((x + y)/2, 0) < r$ unless $x = y$; $y \in E$ and r is any positive real number. We prove that a metric linear space (E, d) is strictly convex if and only if all convex subsets of E are semi-Chebyshev.

The notion of strict convexity in normed linear spaces was extended to metric linear spaces in [**1**] and a characterization of strictly convex metric linear spaces vis. "A metric linear space is strictly convex if and only if its convex proximinal sets are Chebyshev" was proved in [**4**]. For strictly convex normed linear spaces this characterization was proved by Phelps [**5**]. Another characterization of strictly convex normed linear spaces (see e.g. [**3**]) viz "A normed linear space is strictly convex if and only if its convex subsets (linear subspaces) are semi-Chebyshev" is well known. We shall show, together with some results that a similar characterization of strictly convex metric linear spaces is true.

We start with a few definitions. Let G be a subset of a metric linear space (E, d) and $x \in E$. An element $g_0 \in G$ is said to be a *best approximation* to x in G if $d(x, g_0) = d(x, G)$. The set G is said to be *proximinal* (respectively *semi-Chebysev*), if each element of E has at leaost one (respectively at most one) best approximation in G. G is said to be *Chebyshev* if it is proximinal as well as semi-Chebyshev. A mapping f which takes each element x of E to its set of best approximations in G is called the *metric projection* or the *nearest point map* or the *best approximation map*.

A metric linear space (E, d) is said to have *property* (P) if the nearest point map shrinks distances whenever it exists.

A metric linear space (E, d) is said to have *property* (P_1) if for every pairof elements $x, z \in E$ such that $d(x+z, 0) \leq d(x, 0)$ there exist constants $b = b(x, z) > 0$ $c = c(x, z) > 0$ such that $d(y + cz, 0) \leq d(y, 0)$ for $d(y, x) \leq b$.

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A metric linear space (E, d) is said to be *strictly convex* if $d(x, 0) \leq r$, $d(y, 0) \leq r$ imply $d((x+y)/2, 0) < r$ unless $x = y$; $x, y \in E$ and r is any positive real number.

First we shall show (Theorem 1) that strict convexity is weaker tha property (P) but stronger than property (P_1). We shall need the following two lemmas in the sequel.

LEMMA 1. *In a metric linear space (E, d) the line segment $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$ is a compact convex set.*

For proof of this we refer to Lemma 1 of [**4**].

LEMMA 2. *Let (E, d) be a metric linear space. Then the following statements are equivalent:*

(i) *$r > 0$, $d(x, 0) = r = d(y, 0)$ and $x \neq y$ imply $B(0, r) \cap]x, y[= \emptyset$.*

(ii) *(E, d) is strictly convex.*

(iii) *$r > 0$, $\neq xy$, $x, y \in B[0, r]$ imply $]x, y[\subset B(0, r)$.*

Here $]x, y[= \{tx + (1-t)y : 0 < t < 1\}$, $B[0, r] = \{z \in E : d(z, 0) \leq r\}$ and $B(0, r) = \{z \in E : d(z, 0) < r\}$.

For proof of this lemma we refer to [**6**, Theorem 1.8.]

THEOREM 1. *Let (E, d) be a metric linear space. We have*

(i) *If (E, d) has property (P) then it is strictly convex.*

(ii) *If (E, d) is strictly convex then it has property (P_1).*

Proof. (i) Suppose (E, d) is not strictly convex. Then by Lemma 2, there exists an $r > 0$ and distinct points x and y such that $d(x, 0) = d(y, 0) = r$ and $B(0, r) \cap]x, y[= \emptyset$. Consider the compact line segment $[x, y]$. This set is proximinal. Let $f : E \to [x, y]$ be the nearest point map. Then $f(0) = x$, $f(0) = y$. Consider

$$d(x, y) = d(f(0), f(0)) \leq d(0, 0) = 0 \quad \text{[by ProPerty } (P)],$$

and so $x = y$, a contradiction.

(ii) If $d(x + z, 0) < d(x, 0)$ and $2d(y, x) \leq d(x, 0) - d(x + z, 0)$ then

$$d(y + z, 0) \leq d(x + z, 0) - d(y, x) \leq d(y, 0)$$

Thus property (P_1) is satisfied if $b[d(x, 0) - d(x + z, 0)]/2$ and $c = 1$.

If $d(x + z, 0) < d(x, 0)$ then by the strict convexity,

$$d(x + z/2, 0) = d((x + z + x)/2, 0) < d(x, 0)$$

and so property (P_1) is satisfied if $b = [d(x, 0) - d(x + z/2, 0)]/2$ and $c = 1/2$ as

$$\begin{gathered} d(y + z/2, 0) \leq d(y, x) + d(x + z/2, 0) = \\ = d(y, x) + d(x, 0) - 2b \leq d(x, 0) - b \leq d(y, 0). \end{gathered}$$

Remark. In normed linear spaces, the first part of this theorem was proved in [**5**] and second part in [**2**].

The following theorem gives a characterization of strictly convex metric linear spaces.

THEOREM 2. *A metric linear space (E, d) is strictly convex if and only if all convex subsets of E are semi-Chebyshev.*

it Proof. Let (E, d) be strictly convex and G be a convex subset of E.

Suppose there exists some $x \in E|G$ which has two distinct best approximations in G, say g_1 and g_2 i.e. $d(x, g_1) = d(x, g_2) = d(x, G)$. Then by the strict convexity, $d(x, (g_1 + g_2)/2) < d(x, G)$, a contradiction as $(g_1 + g_2)/2 \in G$. Therefore G must be semi-Chebyshev.

Conversely, suppose all convex subsets of the metric linear space (E, d) are semi-Chebyshev. Suppose (E, d) is not strictly convex. Then by Lemma 2, there exists an $r > 0$ and distinct points $x, y \in E$ such that $d(x, 0) = d(y, 0) = r$ and $B(0, r) \cap]x, y[= \emptyset$. Consider the convex line segment $[x, y]$. It is not semi-Chebyshev since for the point 0 of E there are two distinct best approximations (x and y), a contradiction.

Remark. Replacing the line segment $[x, y]$ by the real one-dimensional subspace $G = \{\lambda(y - x) : -\infty < \lambda < \infty\}$ in the second part of the proof of the above theorem we can see that G is not semi-Chebyshev as for the element $-x \in E$, both 0 and $y - x$ are best approximations in G and hence it follows that a metric linear space (E, d) is strictly convex if and only if linear subspaces of E are semi-Chebyshev.

REFERENCES

[1] G. C. Ahuja, T. D. Narang and Swaran Trehan, *Best Approximution on convex sets in metric linear spaces*, Math. Nachr. **78** (1977), 125–130.

[2] A. L. Brown, *Best n-dimensional approximation to sets of functions*, Proc. London Math. Soc. **14** (1964), 577–594.

[3] R. A. Hirschfeld, *On best approximations in normed vector spaces*, Nieuw Arch. Wisk. **6** (1958), 41–51.

[4] T. D. Narang, *Unicity theorem und strict convexity of metric lineur spures*, Tamkang J. Math. **11** (1980), 49–51; Corrigendum, **14** (1983), 103–l04.

[5] R. R. Phelps, *Convex sets and nearest points*, Proc. Amer. Math. Soc. **8** (1975), 790–797.

[6] K. P. R. Sastry and S. V. R. Naidu, *Convexity conditions in metric linear spaces*, Math. Seminar Notes **7** (1979), 235–251.

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