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THE STRUCTURE ON A SUBSPACE OF A SPACE WITH AN $f(3,-1)$-STRUCTURE

Jovanka Nikić

Abstract. Let $\mathcal{M}^n$ be a manifold with an $f(3,-1)$-structure of rank r and let $\mathcal{N}^{n-1}$ be a hypersurface in $\mathcal{M}^n$. The following theorem is proved: If the dimension of $T(\mathcal{V}^{n-1} \cap f(T\mathcal{N}^{n-1}))_p$ is constant, say s, for all $p \in \mathcal{N}^{n-1}$, then $\mathcal{N}^{n-1}$ possesses a natural $F(3,-1)$-structure of rank s. It is also proved that the naturally induced $F(3,-1)$-structure is integrable if the $f(3,-1)$-structure on $\mathcal{M}^n$ is integrable and if the transversal to $\mathcal{N}^{n-1}$ can be found to lie in the distribution M.

Introduction. Let $\mathcal{M}^n$ be an n-dimensional C^∞ manifold and f a tensor of type (1,1) such that $f^3 \pm f = 0$, and let the rank of f be constant, say $r < n$, on $\mathcal{M}^n$. We then say that $\mathcal{M}^n$ has an $f(3,\pm 1)$ structure of rank r.

Yano and Ishihara [**5**] have shown that if $\mathcal{M}^n$ is an almost complex manifold, then a submanifold of $\mathcal{M}^n$ satisfying a certain property, possesses a natural $f(3,+1)$-structure. In particular, Tashiro [**4**] has shown that if the submanifold is a hypersurface, then the induced $f(3,+1)$-structure has a maximal rank (i.e. is an almost contact structure).

It is known that a hypersurface of an almost contact manifold possesses a natural $f(3,+1)$-structure, which does not have a maximal rank. In [**2**] Miyzawa proved that if the submanifold in the manifold with an almost product structure is a hypersurface, then the induced $f(3,-1)$-structure has a maximal rank. (i.e. is an almost paracontact structure.)

The purpose in this paper is to show that if $\mathcal{M}^n$ has an $f(3,-1)$-structure, then a hypersurface of this manifold possesses a natural $F(3,-1)$-structure. In section 3 we shall study the integrability of the induced $f(3,-1)$-structure.

Prelimininaries. The (ψ,ξ,η)-structure is an almost paracontact structure if we have on $\mathcal{M}^n$ a tensor field ψ of type (1,1), a vector field ξ, and a 1-form η satisfying:

$$\psi^2 = I - \eta \otimes \xi, \quad \eta\psi = 0, \quad \psi\xi = 0, \quad \eta(\xi) = 1.$$

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In [**2**] the following theorem is proved:

THEOREM 1.1. *Let* $T(\mathcal{M}^n)_p$ *denote the tangent space to* $\mathcal{M}^n$ *at* p. *The almost product structure* Φ *induces the almost paracontact structure* ψ *on a hypersurface* $\mathcal{K}^n$ *in the following way:* $\Phi B = B\psi \oplus (\eta \oplus N)$, $\Phi N = B\xi$, *where* B *is the differential of the immersion* i $\mathcal{K}$ *into* $\mathcal{M}^n$, *and* $N \in (\mathcal{M}^n)_p$ $N \notin T(\mathcal{K})_p$, *for all* $p \in \mathcal{K}$.

THEOREM 1.2. *If is an* $f(3,-1)$*-structure on ,* $\mathcal{M}^n$ *the operators* $l = f^2$, $m = I - f^2$, *I denoting the identity operator, applied to the tangent space at a point of the manifold, are complementary projection operators.*

There exist complementary distributions L and M corresponding to the projection operators l and m, respectively.

2. The structure on the hypersurface. THEOREM 2.1 *Let* $\mathcal{M}^n$ *be a manifold with* $f(3,-1)$*-structure of rank* r *and let* $\mathcal{N}^m$ *be a hypersurface in* $\mathcal{M}^n$, $m = n-1$. *It the dimnsion of* $T(\mathcal{N}^m)_p \cap f(T(\mathcal{N}^m))_p$ *is constant, say* s, *for all* $p \in \mathcal{N}^m$, *then* $\mathcal{N}^m$ *possesses a natural* $F(3,-1)$*-structure of rank* s.

Proof. Let C be a transversal defined on $\mathcal{N}^m$, i.e. $C \in T\ (\mathcal{M}^n)_p$ but $C \notin T(\mathcal{N}^m)_p$ for all $p \in \mathcal{N}^m$. Let B be a differential of the imbedding of $\mathcal{N}^m$ in $\mathcal{M}^n$. Then B is a map of $T(\mathcal{N}^m)$ into $T_R(\mathcal{M}^n)$, where $T_r(\mathcal{M}^n)$ denotes the restriction of $T(\mathcal{M}^n)$, the tangent bundle of $\mathcal{M}^n$ to $\mathcal{N}^m$. Then we can find a locally 1-form C^* defined on $\mathcal{N}^m$ such that:

$$\begin{aligned} &B^{-1}B = I, && BB^{-1} = I - C^* \otimes C,\\ &C^*B = B^{-1}C = 0, && C^*(C) = 1 \quad . \end{aligned}$$

Let F be defined locally on $T(\mathcal{N}^m)$ by $F = B^{-1}fB$. Then:

$$\begin{aligned} &F^2X = B^{-1}fBB^{-1}fBX = B^{-1}f(I_C^* \otimes C)f(BX) =\\ &=B^{-1}f^2(BX) - C^*f(BX)B^{-1}fC. \end{aligned}$$

If C is in distribution M, then $fC = 0$ we have that

$$\begin{aligned} &(F^3 - F)X = B^{-1}fBB^{-1}f^2BX - fBX =\\ &=B^{-1}f(I - C^* \otimes C)f^2BX - B^{-1}fBX = B^{-1}((f^3 - f)Bx) = 0 \end{aligned}$$

fur all X. On the other hand, suppose that C is in distribution L. Then:

$$\begin{aligned} &(F^3 - F)X = (B^{-1}fB)B^{-1}f^2(BX) - (B^{-1}fB)C^*(fBX)B^{-1}fC -\\ &-B^{-1}fBX = B^{-1}(f^3 - f)BX - C^*(f^2BX)B^{-1}fC -\\ &-C^*(fBX)^{-1}f^2C + C^*(fBX)C^*(fC)B^{-1}fC = 0, \end{aligned}$$

since $f^2C = C$ on L and $C^*B = B^{-1}C = 0$, and since we can choose C^* so that $C^*(fC) = 0$. Also $C^*(f^2BX) = C^*(BX + (f^2 - 1)BX) = 0$.

THEOREM 2.2. *If* (f, ξ, η) *is an almost paracontact structure on* $\mathcal{M}^n$ *then* $\mathcal{N}^{-1}$ *possesses a natural* $F(3,-1)$*-structure if* ξ *is tangent to* $\mathcal{N}^{n-1}$. *The hypersurface* $\mathcal{N}^{n-1}$ *possesses a natural almost product structure if 5 is not tangent to* $\mathcal{N}^{n-1}$.

Proof. When ξ is not tangent to $\mathcal{N}^{-1}$, ξ can be chosen for a pseudonormal. Then we have from Theorem 2.1. that $T(\mathcal{N}^{-1}) \cap f(T(\mathcal{N}^{-1})) = T(\mathcal{N}^{-1})$, and rank $F = \dim \mathcal{N}^{-1} = n-1$. The almost paracontact structure F has a maximal rank, i.e. F is an almost product structure.

3. Integrability conditions. The structure f is integrable if $[f,f]=0$; $[f,f]$ denotes the Nijenhuis tensor of f, i.e.

$$[f,f](X,Y) = [fX,fY) - [fX,Y] - f(X,fY) + f^2[X,Y]$$

for all vector fields X an Y on $\mathcal{M}^n$.

In this section we shall assume that f is an $f(3,-1)$-structure on $\mathcal{M}^n$ and that $F = B^{-1}fB$ is the naturally induced $F(3,-1)$-structure.

THEOREM 3.1. *Let $\mathcal{N}^{-1}$ be a hypersurface in $\mathcal{M}^n$, and suppose f is integrable. If, locally, the transversal to $\mathcal{N}^{-1}$ can be found to lie in tbe distribution M, then the induced $F(3,-1)$-structure on $\mathcal{N}^{-1}$ is integrable.*

Proof. We see that

$$\begin{aligned}
&[F,F](X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y] = \\
=&B^{-1}[BB^{-1}fBX, BB^{-1}fBY] - B^{-1}f[BB^{-1}fBX, BY] - \\
-&B^{-1}f[BX, BB^{-1}fBY] + B^{-1}fBB^{-1}f[BX,BY] = \\
=&B^{-1}\{[f,f](BY,BY) - [C^*(fBX)C, fBY] - [fBX, C^*(fBY)C] + \\
+&[C^*(fBX)C, C^*(fBY)C] + f[C^*(fBX)C, BY] + f[BX, C^*(fBYC] - \\
-&C^*(f[BX,BY])fC\},
\end{aligned}$$

where we have used the fact that $B[X,Y) = [BX,BY]$, for vector fields X, Y on $\mathcal{N}^{-1}$, and that $BB^{-1} = I - C^* \otimes C$. If the transversal C lies in the distribution M, then the form C^* can be chosen so that $C^*f = 0$. We see that: $[F,F](X,Y) = B^{-1}([f,f](BX,BY)) = 0$.

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Fakultet tehničkih nauka
Institut za primenjene osnovne discipline
Veljka Vlahovića 3
21000 Novi Sad
Jugoslavija

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