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# SOME GLOBAL PROPERTIES OF PLANE CURVES

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**Abstract.** We introduce $L$-involutions for any positive number $L$ and we give a characterization of the class $(L)$ of all $L$-involutions. Then we define so-called $\nu$-involutive pairs of points of a curve $C \in \mathcal{M}$ where $\mathcal{M}$ is the family of all $C^1$ plane closed curves. For arbitrary $C \in \mathcal{M}$ of length $L$ and for arbitrary $\nu \in (L)$ there exists a $\nu$-involutive pair of $C$ such that the tangent lines at the points ot this pair are parallel. Applications of this fact are given.

We describe some results belonging to the global differential geometry of curves. We denote by $\mathcal{M}$ the family of all closed plane curves of class $C^1$ [**1**]. Throughout this paper we assume that all curves of $\mathcal{M}$ are parametrized by arc length.

**1. Preliminaries.** Let $\mathbf{R}$ denote the set of all real numbers and let $J = \{a \in boldR: \ a \geq 0\}$. We introduce the notation $[z, w] = z^1w^2 - z^2w^1$ for elements $z = (z^1, z^2)$, $w = (w^1, w^2)$ of $\mathbf{R}^2$. Let us fix a positive real number $L$.

*Definition* 1. Each function $\nu : J \to \mathbf{R}$ which satisfies the following conditions

1° $0 < \nu(0) < L$,
2° $\nu'(a) > 0$ for all $a > 0$,
3° $\nu(a + L) = \nu(a) + L$ for all $a \geq 0$,
4° $\nu(\nu(a)) = a + L$ for all $a \geq 0$

will be called all $L$-involution.

We denote by $(L)$ the set of all $L$-involutions. It is easy tu see that $\nu$ given by the forlnula $\nu(a) = a + L/2$ is an $L$-involution. Other examples and the geometric meaning of these $L$-involutions will be given later.

We will prove a lemma concerned with the class $(L)$. Let us fix a function $a$ such that

5° $a$ is defined and continuous in $J$,

6° $a(s) > 0$ for all $s \in J$,

7° $a(s + L) = a(s)$ for all $s \in J$.

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Making use the function

$$A(s) = \int_0^s a(u)du$$

we prove the following lemma.

LEMMA. *The solution of the equation $\nu' = a/(a \circ \nu)$ with the initial condition $\nu(0) = A^{-1}(A(L)/2)$ belongs to $(L)$.*

*Proof.* Let us note that the solution $\nu$ is given by the formula

$$\nu(s) = A^{-1}(A(s) + A(L)/2 \qquad \text{for all } \ s \in J$$

The periodicity of the function $a$ implies

$$A(s+L) = \int_0^{s+L} a(u)du = \int_0^s a(u)du + \int_s^{s+L} a(u)du = A(s) + A(L),$$
$$A(s+L) = A(s) + A(L) \qquad \text{for all} s \in J.$$

Using the above formulas we can verify the conditions 3°, 4°.

$$\begin{aligned}\nu(\nu(s)) &= A^{-1}(A(\nu(s)) + A(L)/2) = A^{-1}(A(s) + A(L)) = A^{-1}(A(s+L)) = \\ &= s + L, \\ A(\nu(s) + L) &= A(\nu(s)) + A(L) = A(s) + A(L)/2 + A(L) = A(s+L) + A(L)/2 = \\ &= A(\nu((s+L)).\end{aligned}$$

The conditions 1°, 2° are trivial.

**Main result.** Let us fix $C \in \mathcal{M}$, $s \mapsto x(s) = (x^1(s)) \in \mathbf{R}^2$ for $0 \leq s \leq L$ and $\nu \in (L)$.

*Definition.* 2 A pair of points $(x(s), x(\nu(s)))$ of $C$ will be called $\nu$-involutive pair.

THEOREM 1. *Let $C \in \mathcal{M}$ and let $L$ be its length. For arbitrary $\nu \in (L)$ there exists a $\nu$-involutive pair $C$ such that the tangent lines at points of this pair are parallel.*

*Proof.* Let $C \in \mathcal{M}$, $s \mapsto x(s)$ for $0 \leq s \leq L$; we prolongate the functions $x^1$, $x^2$ periodically onto $J$.

Let us note that the function $\vartheta(s) = [x'(s), x'(\nu(s))]$ satisfies the conditions $\vartheta(0) = \vartheta(L)$ and

$$\vartheta \circ \nu = -\vartheta. \tag{1}$$

We prolongate the function $\vartheta$ periodically onto $J$. Making use of (1) we obtain

$$\int_0^L (\nu'(s)+1)\vartheta(s)ds = -\int_0^L \nu'(s)\vartheta(\nu(s))ds + \int_0^L \vartheta(s)ds =$$
$$= -\int_{\nu(0)}^{\nu(0)+L} \vartheta(u)du + \int_0^L \vartheta(s)ds = 0$$

and

$$\int_0^L (\nu'(s)+1)\vartheta(s) = 0. \tag{2}$$

Thus there exists $0 \leq u < L$ such that $\vartheta(u) = 0$, finishing the proof.

**3. Applications.** Let us consider $C \in \mathcal{M}$ of the length $L$. Using the $L$-involution $\nu(a) = a + L/2$ we obtain the following corollary to Theorem 1.

THEOREM 2. *For an arbitrary curve of the family* $\mathcal{M}$ *there exists a pair of its points such that:* 1° *the tangent lines at these points are parallel;* 2° *these points divide the length of the curve into equal parts.*

Now we give another application of Theorem 1. To do this, let us consider a simple, convex curve $C \in \mathcal{M}$ [**1**]. Without loss of generality we can assume that the origin lies in the interior of $C$. Let $C : s \mapsto x(s)$ for $0 \leq s \leq L$, and let $0 < a < L$ be chosen so that the segment joining the points $x(0)$ and $x(a)$ divides the area of the region bounded by $C$ into equal parts. Let us consider the differential equation

$$\beta' = \frac{[x - x \circ \beta, x']}{[x \circ \beta - x, x' \circ \beta]} \tag{3}$$

with the initial condition $\beta(0) = a$.

Now we investigate the differences of areas of regions bounded by the segment joining the points $x(s)$, $x(\beta(s))$ and hy arcs of $C$ contained between these points.

Using differential equation (2) we get

$$\left\{ \left( \frac{1}{2} \int_s^{\beta(s)} [x(u), x'(u)]du - \frac{1}{2}[(u, x(\beta(u))] \right) - \right.$$
$$\left. - \left( \frac{1}{2} \int_{\beta(s)}^{s+L} [x(u), x'(u)]du + \frac{1}{2}[x(u), x(\beta(u))] \right) \right\}' =$$
$$= \beta'(s)[x(\beta(s)), x'(\beta(s))] - [x(s), x'(s)] - [x'(s), x(\beta(s))] - [x(s), x'(\beta(s))\beta'(s)] =$$
$$= \beta'(s)[x(\beta(s)) - x(s), x'(\beta(s))] - [x(s) - x(\beta(s)), x'(s)] = 0.$$

Thus the segment joining the points $x(s)$, $x(\beta(s))$ divides the area of the region bounded by $C$ into equal parts. The geometric interpretation of $\beta$ is clear. So $\beta$ is an $L$-involution and we have

THEOREM 3. *For all arbitrary simple, convex curve $C$ of the family $\mathcal{M}$ there exists a pair of its points such that* $1^\circ$ *the tangents lines of these points are parallel;* $2^\circ$ *the segment joining these points divides the area of the region bounded by $C$ into equal parts.*

At the end we give an example of $L$-involution considered in [**2**].

Let $C \in \mathcal{M}$ be an oval with positive curvature $k$ and perimeter $L$. The solution $\psi$ of the differential equation $\psi' = k/(k \circ \psi)$ with a corresponding intial condition is an $L$-involution. The tangent lines at points of an arbitrary $\psi$-involutive pair are parallel.

## REFERENCES

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