PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 40 (54), 1986, pp.11–15

ON A QUASIORDERING OF BIPARTITE GRAPHS

Ivan Gutman and **Zhang Fuji**

Abstract. A quesiordering of bipartite graphs, based on the coefficients of their characteristic polynomials, is considered. Three novel statements are deduced which generalize certain previous results [5–7] of one of the present authors.

1. Introduction and the Main Result

Let G be a bipartite graph on p vertices. It is well known [1] that the characteristic polynomial of G can be presented in the form

$$\Phi(G) = \Phi(G, x) = \sum_{k=0}^{[p/2]} (-1)^k b(G, k) x^{p-2k}$$

where $b(G, 0) = 1$ and $b(G, k) \geq 0$ for all $k \geq 1$. For two bipartite graphs G and H (which need not posses equal number of vertices), we define a relation $G \succ H$ as $b(G, k) \geq b(H, k)$ for all $k \geq 1$. If $G \succ H$ and $H \succ G$, then we write $G \sim H$.

The relation $\succ$ induces a quasiordering on the set of all bipartite graphs. This quasiordering has been introduced by one of the present authors [2, 3]. A number of results, concerning the relation $\succ$, has been recently obtained [5, 6, 7]. In the present work we communicate a few additional findings of the same type, which, in particular, generalize certain previously known results.

The significance of the quasiordering $\succ$ lies in the following. Let $x_1, x_2, \dots, x_p$ be the eigenvalues of the graph G. Then the quantity $E(G) = |x_1| + |x_2| + \cdots + |x_p|$ is called the energy of the graph G. It has been demonstrated [3] that if G and H are bipartite and $G \succ H$, then $E(G) \geq E(H)$. It is worth mentioning that the energy of a graph plays an important role in theoretical chemistry. (For review of the chemical applications of $E(G)$ see [4].).

The results of the present work are summarized in the following three theorems.

AMS Subject Classification (1980): Primary 05C50.

Let P_n and C_n denote the path and the cycle, respectively, with n vertices. Let their vertices $\nu_1, \nu_2, \ldots, \nu_n$ be labeled so that ν_i and ν_{i+1} are adjacent for $i = 1, \ldots, n-1$. In addition, the vertices ν_1 and ν_n of C_n are also adjacent.

If two graphs G and H are isomorphic, we shall write $G = H$.

Let G be a graph and u and ν its two vertices. The subgraph obtained by deletion of the vertex ν (respectively u) from G will be denoted by G_ν (respectively G_u). We say that the vertices u and ν are equivalent if $G_u = G_\nu$.

Let H be another graph, w its vertex and H_w the subgraph obtained by deletion of the vertex w from H.

Denote by $G(\nu, w)\ H$ the graph obtained by coalescing the vertices ν and w of G and H, respectively. In particular, $P_n(r, \nu)\ G$ is the graph obtained from P_n and G by identifying the vertex ν_r of P_n with the vertex ν of G. We denote by $G(\nu, r)\, C_n(s, w)\, H$ the graph obtained by coalescing the vertex ν of G and the vertex ν_r of C_n, and by coalescing the vertices ν_s of C_n and w of H. Without loss of generality we may assume that $r = 1$ and that $s \leq [n/2] + 1$.

Let u and ν be two distinct vertices of G. Denote by $S_a(u)\, G(\nu)\, S_b$ the graph obtained by attaching a new vertices of degre one to the vertex u, and b new vertices of degree one to the vertex ν of G.

Proposition 1. *If G is bipartite and ν is its arbitrary vertex, then*

(a)
$$\begin{aligned} P_n(1,\nu)G \succ P_n(3,\nu)G \succ \cdots \succ P_n(2k-1,\nu)G \succ P_n(2k,\nu)G \\ \succ P_n(2k-2,\nu)G \succ \cdots \succ P_n(2,\nu)G \end{aligned}$$

for $n = 4k-1$ or $n = 4k$, and

(b)
$$\begin{aligned} P_n(1,\nu)G \succ P_n(3,\nu)G \succ \cdots \succ P_n(2k+1,\nu)G \succ P_n(2k,\nu)G \\ \succ P_n(2k-2,\nu)G \succ \cdots \succ P_n(2,\nu)G \end{aligned}$$

for $n = 4k+1$ or $n = 4k+2$.

In the special case when G is a star and ν is its central vertex, the above statement reduces to Theorem 1 from [7].

Proposition 2. *If G and H are bipartite graphs and n is even, then for arbitrary vertices ν and w,*

(a)
$$\begin{aligned} G(\nu,1)C_n(2,w)H \succ G(\nu,1)C_n(4,w)H \succ \cdots \succ G(\nu,1)C_n(2k,w)H \\ \succ G(\nu,1)C_n(2k+1,w)H \succ G(\nu,1)C_n(2k-1,w)H \\ \cdots \succ G(\nu,1)C_n(3,w)H \end{aligned}$$

for $n = 4k$, and

(b)
$$\begin{aligned} G(\nu,1)C_n(2,w)H \succ G(\nu,1)C_n(4,w)H \succ \cdots \succ G(\nu,1)C_n(2k+2,w)H \\ \succ G(\nu,1)C_n(2k+1,w)H \succ G(\nu,1)C_n(2k-1,w)H \\ \cdots \succ G(\nu,1)C_n(3,w)H \end{aligned}$$

for $n = 4k + 2$.

In the special case when both vertices ν and w have degree one, Proposition 2 reduces to Corollary 2 of [5].

PROPOSITION 3. *If G is bipartite and its two vertices u and ν are equivalent, then*

$$S_m(u)\,G(\nu)\,S_0 \prec S_{m-1}(u)\,G(\nu)\,S_1 \prec \cdots \prec S_{m-[m/2]}(u)\,G(\nu)\,S_{[m/2]}.$$

A special case of Proposition 3, namely when G is a path and u and ν are its terminal vertices, is just Theorem 2 from [7].

2. Preliminaries

In order to prove Proposition 1-3 we need some preparations. In what follows G and H denote bipartite graphs. The graph whose components are G and H is denoted by $G\dot{+}H$.

Let E_n be the graph with n vertices and without edges. Since [1]

$$\Phi(G\dot{+}E_n) = x^n\,\Phi(G),$$

we have the following simple result.

LEMMA 1. $G\dot{+}E_n \sim G.\square$

Without proof we refer to the following three previously known statements.

LEMMA 2 [2]. (a) *If $n = 4k$ or $4k + 1$, then*

$$P_n \succ P_2\dot{+}P_{n-2} \succ \cdots \succ P_{2k}\dot{+}P_{n-2k} \succ P_{2k-1}\dot{+}P_{n-2k+1}$$
$$\succ P_{2k-3}\dot{+}P_{n-2k+3} \succ \cdots \succ P_1\dot{+}P_{n-1}.$$

(b) *If $n = 4k + 2$ or $4k + 3$, then*

$$P_n \succ P_2\dot{+}P_{n-2} \succ \cdots \succ P_{2k}\dot{+}P_{n-2k} \succ P_{2k+1}\dot{+}P_{n-2k-1}$$
$$\succ P_{2k-1}\dot{+}P_{n-2k+1} \succ \cdots \succ P_1\dot{+}P_{n-1}.$$

LEMMA 3 [1].

$$\Phi(G(\nu,w)H) = \Phi(G)\,\Phi(H_w) + \Phi(G_\nu)\,\Phi(H) - x\Phi(G_\nu)\,\Phi(H_w)$$

i.e.

$$b(G(\nu,w)H,\,k) = b(G\dot{+}H_w,\,k) + b(G_\nu\dot{+}H,\,k) - b(G_\nu\dot{+}H_w,\,k).$$

LEMMA 4 [1]. *Let ν be a vertex of G having degree one and being adjacent to the vertex u. Let G_{uv} denote the graph obtained by deleting both u and ν from G. Then*

$$\Phi(G) = x\Phi(G_\nu) - \Phi(G_{u\nu}) \text{ i.e. } b(G,k) = b(G_w,k) + b(G_{u\nu},k-1).$$

LEMMA 5. *If G is bipartite and ν is its vertex, then $G \succ G_\nu$.*

Proof. Let $x_1 \geq x_2 \geq \cdots \geq x_p$ be the eigenvalues of G and $y_1 \geq y_2 \geq \cdots \geq y_{p-1}$ the eigenvalues of G_ν, where p is the number of vertices of G. Since G is bipartite [1],

$$x_k + x_{p-k+1} = 0 \quad \text{and} \quad y_k + y_{p-k} = 0 \quad \text{for} \quad k = 1, 2, \dots, [p/2].$$

Consequently

$$\Phi(G, x) = x^{p-2q} \Pi_{j=1}^{q} (x^2 - x_j^2)$$

and

$$\Phi(G_\nu, x) = x^{p-2q-1} \Pi_{j=1}^{q} (x^2 - y_j^2),$$

where q is chosen so that $x_q > 0$ and $x_{q+1} \leq 0$. It is now immediate that

$$b(G, k) = \sum_{j_1 < j_2 < \cdots < j_k \leq q} x_{j_1}^2 x_{j_2}^2 \cdots x_{j_k}^2,$$

$$b(G_\nu, k) = \sum_{j_1 < j_2 < \cdots < j_k \leq q} y_{j_1}^2 y_{j_2}^2 \cdots y_{j_k}^2.$$

Lemma 5 follows now from the Cauchy interlacing [1], viz.,

$$x_1 \geq y_1 \geq x_2 \geq y_2 \cdots \geq x_{p-1} \geq y_{p-1} \geq x_p. \square$$

Proof of Proposition 1. Applying Lemma 3 to $P_n(r, \nu)G$ one obtains

$$b(P_n(r, \nu)G, k) = b(P \dot{+} G_\nu, k) + b(P_{r-1} \dot{+} P_{n-r} \dot{+} G, k) - b(P_{r-1} \dot{+} P_{n-r} \dot{+} G_\nu, k).$$

Note that $b(P_n \dot{+} G_\nu, k)$ is independent of the variable r. Having in mind that because of $\Phi(G \dot{+} H) = \Phi(G)\Phi(H)$,

$$b(G \dot{+} H, k) = \Sigma_j b(G, j) b(H, k - j),$$

we conclude that

$$b(P_n(r, \nu)G, k) = b(P_n \dot{+} G_\nu, k) + \Sigma_j b(P_{r-1} \dot{+} P_{n-r}, j)[b(G, k - j) - b(G_\nu, k - j)].$$

Since by Lemma 5, $b(G, k-j) - b(G_\nu, k-j) \geq 0$ for all values of $k - j$, it is evident that $P_n(r, \nu)G \succ P_n(s, \nu)G$ if and only if $P_{r-1} \dot{+} P_{n-r} \succ P_{s-1} \dot{+} P_{n-s}$. The rest of the proof is now straightforward from Lemma 2.$\square$

Proof of Proposition 2. Applying Lemma 3 to $G(\nu, 1)C_n(s, w)H$ one obtains

$$\begin{aligned} b(G(\nu, 1)C_n(s, w)H, k) &= b(G(\nu, 1)C_n \dot{+} H_w, k) \\ &+ b(P_{n-1}(s-1, \nu)G \dot{+} H, k) - b(P_{n-1}(s-1, \nu)G \dot{+} H_w, k). \end{aligned}$$

Again, the first term on the right-hand side is independent of the parameter s. Using the same argument as before we deduce that

$$G(\nu, 1)C_n(s, w)H \succ G(\nu, 1)C_n(t, w)H$$

if and only if

$$P_{n-1}(s-1,\nu)G \succ P_{n-1}(t-1,\nu)G.$$

Proposition 2 follows now from Proposition 1.□

Proof of Proposition 3. Consider a vertex of degree one of $S_a(u)G(\nu)S_b$, which is attached to the vertex u of G. Consider a vertex of degree one of $S_{a-1}(u)G(\nu)S_{b+1}$, attached to the vertex ν of G. Applying Lemma 4 to these two vertices one gets.

$$b(S_a(u)G(\nu)S_b,k) = b(S_{a-1}(u)G(\nu)S_b,k) + b(E_{a-1}\dot{+}G_u(\nu)S_b,k-1)$$
$$b(S_{a-1}(u)G(\nu)S_{b+1},k) = b(S_{a-1}(u)G(\nu)S_b,k) + b(E_b\dot{+}G_\nu(u)S_{a-1},k-1).$$

Here $H(w)S_a$ denotes the graph obtained by attaching a vertices of degree one to the vertex w of H.

Comparing the above relations and having in mind Lemma 1, one concludes that $S_a(u)G(\nu)S_b \succ S_{a-1}(u)G(\nu)S_{b+1}$ if and only if $G_u(\nu)S_b \succ G_\nu(u)S_a$. On the other hand, if the vertices u and ν are equivalent, then $G_\nu(u)S_{a-1}$ is a subgraph of $G_u(\nu)S_b$ whenever $b \geq a-1$. This means (because of Lemma 5) that for $b > a$, $S_a(u)G(\nu)S_b \succ S_{a-1}(u)G(\nu)S_{b+1}$.

Proposition 3 follows now immediately.□

Let $[S_a(u)G(\nu)S_b](u,w)H$ be the graph obtained by coalescing the vertices u of $S_a(u)G(\nu)S_b$ and w of H. The graph $[S_a(u)G(\nu)S_b](\nu,w)H$ is defined analogously. Then a direct application of Lemmas 3 and 5 leads to the following enhancement of Proposition 3.

COROLLARY 3.1. *If the conditions of Proposition 3 are fulfilled and if H is bipartite, then $[S_a(u)G(\nu)S_b](u,w)H \succ [S_a(u)G(\nu)S_b](\nu,w)H$ whenever $a \leq b$.*

REFERENCES

[1] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980.

[2] I. Gutman, *Acyclic systems with extremal Hückel π-electron energy*, Theoret. Chim. Acta **45** (1977), 79–87.

[3] I. Gutman, *Partial ordering of forests according to their characteristic polynomials*, in: A. Hajnal and V. T. Sós (Eds.). *Combinatorics*, North-Holland, Amsterdam 1978, 429–436.

[4] I. Gutman, *Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology*, Bull. Soc. Chim Beograd **43** (1978), 761–774.

[5] Zhang Fuji, *Two theorems of comparison of bipartite graphs by their energy*, Kexue Tongbao **28** (1983), 726–730.

[6] Zhang Fuji, Lai Zaikang, *The totally ordered subset in a sort of Hasse graphs of trees*, J. Xinjiang Univ. (1983), 12–20.

[7] Zhang Fuji, Lai-Zaikang, *Three theorems of comparison of trees by their energy*, Science Exploration **3** (1983), 35–36.

Prirodno-matematički fakultet
34000 Kragujevac
Yugoslavia

(Received 28 01 1985)

Department of Mathematics
Xinjiang University
Urumchi, China