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ON THE GENERAL SOLUTION OF A FUNCTIONAL EQUATION CONNECTED TO SUM FORM INFORMATION MEASURES ON OPEN DOMAIN-I

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Abstract. This paper is devoted to the study of a functional equation connected with the characterization of several information measures. We find the general solution of the functional equation (2) on an open domain, without using 0-probability and 1-probability.

1. Introduction. Let $\Gamma_n^0 = \{P = (p_1, p_2, \dots, p_n) \mid 0 < p_k < 1, \sum_1^n p_k = 1\}$ and Γ_n be the closure of Γ_n^0, that is, $\Gamma_n = \{P = (p_1, p_2, \dots, p_n) \mid p_k \geq 0, \sum_1^n p_k = 1\}$. Additivity Property of Shannon's entropy [1] leads to sum form functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{f=1}^{m} f(q_j) \tag{1}$$

where $P \in \Gamma_n$, $Q \in \Gamma_m$. Chaundy and McLeod [3] studied the functional equation (1) in connection with the statistical thermodynamics in 1960. Since then, several researchers have investigated the solution of (1) and its generalizations under various regularity conditions on f. One of the generalizations of (1) is the following

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} f(p_i), \tag{2}$$

where $P \in \Gamma_n$, $Q \in \Gamma_m$ and α, $\beta \in \mathbf{R} - \{0\}$ (non-zero real numbers). The functional equation (2) was studied in [4,5] under different regularity conditions on f. The general solution of (2) can be found in [9].

In all these above cited papers, the functional equations (1) and (2) were solved with the use of boundary substitution, that is, using 0-probability and 1-probability. The use of these extreme values of the probabilities makes the equations easily solvable. However, the use of these requires definions like $0^{\beta} = 0$ and $0 \log 0 =$

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0. It is also a priori quite possible that there may exist solutions other than those on $[0,1]$ restricted to $]\,0,1\,[$ as shown in [2] for fundamental equation of information, and in [10] for sum form additive functional equations. In this paper, we find the general solution of the functional equation (2) on the open domain adopting the methods found in [6, 8, 9].

2. Solution of (2) on $]\,0,1\,[$. In order to find the general solution of (2) we require the following results.

RESULT 1 [7]. *Let $f_i :]\,0,1\,[\to \mathbf{R}$ be real valued functions and satisfy*

$$\sum_{i=1}^{n} f_i(p_i) = 0, \quad P \in \Gamma_n^0 \text{ (for fixed } n \geq 3)$$

if, and only if $f_i(p) = A(p) + b_i (i = 1, 2, \dots, n)$, $p \in]\,0,1\,[$, where A is additive, b_i are constants with $A(1) + \sum_1^n b_i = 0$.

RESULT 2 [7]. *Let $f :]\,0,1\,[\to \mathbf{R}$ be a real valued function. Then f satisfies $f(pq) = p^\alpha f(q) + q^\alpha f(p)$, for all $p, q \in]\,0,1\,[$ and $\alpha \in \mathbf{R} - \{0\}$ if, and only if, $f(p) = D(p)p^\alpha$ where $D :]\,0,1\,[\to \mathbf{R}$ is a real valued function satisfying*

$$D(pq) = D(p) + D(q), \quad p, q \in]\,0,1\,[, \tag{3}$$

an c is an arbitrary constant.

Now we proceed to find the general solution of (2). Let $f :]\,0,1\,[\to \mathbf{R}$ be a real valued function and satisfy the functional equation (2) for an arbitrary but fixed pair of positive integers m, $n(\geq 3)$, for $P \in \Gamma_n^0$, $Q \in \Gamma_m^0$, with $\alpha, \beta \in \mathbf{R} - \{0,1\}$. Keeping $Q \in \Gamma_m^0$ temporarily fixed in (2) and defining

$$g(p) := \sum_{j=1}^{m} f(pq_j) - p^\alpha f(q_j) - q_j^\beta f(p),$$

and using Result 1, we obtain

$$\sum_{j=1}^{m} [f(pq_j) - p_j^\alpha(q_j) - q_j^\beta f(p)] = A_1^p(p - 1/n, q_1, q_2, \dots, q_m). \tag{4}$$

where $A_1 : \mathbf{R} \times \Gamma_m^0 \to \mathbf{R}$ is an additive function in the first variable. Let $P \in \Gamma_m^0$ and substitute in (4) xp_i for $p (i = 1, 2, \dots, m, x \neq 0)$. Adding the equations so obtained and using (4) again, we get

$$\begin{aligned} \sum_{i=1}^{m} \sum_{j=1}^{m} [f(xp_iq_j) - (p_iq_j)^\beta f(x)] &= x^\alpha \sum_{i=1}^{m} \sum_{j=1}^{m} [p_i^\alpha f(q_j) + q_j^\beta f(p_i)] \\ &+ A_1(x - 1/n, p_1, p_2, \dots, p_m) \sum_{j=1}^{m} q_j^\beta + A_1(x - m/n, q_1, q_2, \dots, q_m). \end{aligned} \tag{5}$$

Since the left hand side of (5) is symmetric in p_i, q_j, we obtain from (5)

$$x^\alpha \sum_{i=1}^{m}\sum_{j=1}^{m}(p_i^\alpha - p_i^\beta) f(q_j) + A_1(x-1/n, q_1, q_2, \dots, q_m) \cdot \sum_{i=1}^{m} p_i^\beta \tag{6}$$
$$- A_1\left(x - \frac{m}{n}, q_1, q_2, \dots, q_m\right) = x^\alpha \sum_{i=1}^{m}\sum_{j=1}^{m}(q_j^\alpha - q_j^\beta) f(p_i)$$
$$+ A_1(x-1/n, p_1, p_2, \dots, p_m) \sum_{j=1}^{m} q_j^\beta - A_1\left(x - \frac{m}{n}, p_1, p_2, \dots, p_m\right).$$

Consider first the case when $\alpha \neq \beta$. Letting $x = n/1$ in (6) and noting the fact that $A_1(0, p_1, p_2, \dots, p_m) = 0$, we get

$$\left(\frac{1}{n}\right)^\alpha \sum_{i=1}^{m}(p^\alpha - p_i^\beta) \sum_{j=1}^{m} f(q_j) - \frac{(1-m)}{n} A_1(1, q_1, q_2, \dots, q_m) \tag{7}$$
$$= \left(\frac{1}{n}\right)^\alpha \sum_{j=1}^{m}(q_j^\alpha - q_j^\beta) \sum_{i=1}^{m} f(p_i) - \frac{(1-m)}{n} A_1(1, p_1, p_2, \dots, p_m).$$

Choose $P^* \in \Gamma_m^0$ such that $\sum_1^m p_i^{*\alpha} - p_i^{*\beta} \neq 0$. Letting $P = P^*$ in (7), we get

$$A_1(1, q_1, q_2, \dots, q_m) = a \sum_{j=1}^{m} f(q_j) + b \sum_{j=1}^{m}(q_j^\alpha - q_j^\beta) + c, \tag{8}$$

where $a(\neq 0)$, b, c are constants. Rewriting (6), we get

$$x^\alpha \sum_{i=1}^{m}(p^\alpha - p_i^\beta) \sum_{j=1}^{m} f(q_j) + \left(\sum_{i=1}^{m} p_i^\beta - 1\right) A_1(x, q_1, q_2, \dots, q_m) \tag{9}$$
$$- \frac{1}{n}(\Sigma p_i^\beta - m) A_1(1, q_1, q_2, \dots, q_m) = x^\alpha \sum_{j=1}^{m}(q_i^\alpha - q_j^\beta) \sum_{i=1}^{m} f(p_i)$$
$$+ \left(\sum_{j=1}^{m} q_j^\beta - 1\right) A_1(x, p_1, p_2, \dots, p_m) - \frac{1}{n}\left(\sum_{j=1}^{m} q_j^\beta - m\right) A_1(1, p_1, p_2, \dots, p_m).$$

As before, keeping $P \in \Gamma_m^0$ fixed such that $\sum_1^m p_i^\beta - 1 \neq 0$ in (9), we get

$$A_1(x, q_1, q_2, \dots, q_m) = A_2(x)\left(\sum_{j=1}^{m} q_j^\beta - 1\right) + a_1 x^\alpha \sum_{j=1}^{m}(q_j^\alpha - q_j^\beta) \tag{10}$$
$$+ b_1\left(\sum_{j=1}^{m} q_j^\beta - m\right) + c_1 x^\alpha \sum_{j=1}^{m} f(q_j) + d_1 A_1(1, q_1, q_2, \dots, q_m),$$

where A_2 is an additive function in reals and a_1, b_1, c_1, d_1 are constants. Since $A_1(x, q_1, q_2, \dots, q_m)$is additive in the first variable, i.e.

$$A_1(x+y, q_1, q_2, \dots, q_m) = A_1(x, q_1, q_2, \dots, q_m) + A_1(y, q_1, q_2, \dots, q_m)$$

and nothing the fact that $\alpha \neq 1$ from (10), we have

$$b_1\left(\sum_{j=1}^{m} q_j^{\beta} - m\right) + d_1 A_1(1, q_1, q_2, \dots, q_m) = 0$$

and

$$a_1 \sum_{j=1}^{m} (q_j^{\alpha} - q_j^{\beta}) + c_1 \sum_{j=1}^{m} f(q_j) = 0,$$

and as a consequence, (10) yields

$$A_1(x, q_1, q_2, \dots, q_m) = A_2(x)(\textstyle\sum_1^m q_j^{\alpha} - 1). \tag{11}$$

Putting $x = 1$ in (11) and comparing with (8), we obtain

$$A_2(1)\left(\sum_{j=1}^{m} q_j^{\beta} - 1\right) = a \sum_{j=1}^{m} f(q_j) + b \sum_{j=1}^{m} (q_j^{\alpha} - q_j^{\beta}) + c \tag{12}$$

Note that $a \neq 0$. Now using Result 1 in (12), we get

$$f(x) = A(x) + a_2 x^{\alpha} + b_2 x^{\beta} + c_2, \quad x \in]\,0, 1\,[. \tag{13}$$

Putting (13) into (2) for $\alpha \neq \beta$, we get $c_2 = 0$, $a_2 = -b_2 = d$ (say) and $A(1) = 0$. Thus (13) becomes (for $\alpha \neq \beta$)

$$f(p) = A(p) + d(p^{\alpha} - p^{\beta}), \quad p \in]\,0, 1\,[\tag{14}$$

where A is an additive function with $A(1) = 0$ and d is an arbitrary constant.

Now we consider the case $\alpha = \beta$. For this case (6) reduces to

$$\begin{aligned} &A_1\left(x - \frac{1}{n}, q_1, q_2, \dots, q_m\right) \sum_{i=1}^{m} p_i^{\alpha} - A_1\left(x - \frac{m}{n}, q_1, q_2, \dots, q_m\right) \\ &= A_1\left(x - \frac{1}{n}, p_1, p_2, \dots, p_m\right) \sum_{j=1}^{m} q_j^{\alpha} - A_1\left(x - \frac{m}{n}, p_1, p_2, \dots, p_m\right). \end{aligned} \tag{15}$$

Letting $x = 1/n$ in (15), we get

$$A_1(1, q_1, q_2, \dots, q_m) = c_3, \text{ constant}. \tag{16}$$

Rewriting (15), we obtain

$$\begin{aligned} &\left(\sum_{i=1}^{m} p_i^{\alpha} - 1\right) A_1(x, q_1, q_2, \dots, q_m) + \frac{c_3}{n}\left(\sum_{j=1}^{m} q_j^{\alpha} - m\right) \\ &= \left(\sum_{j=1}^{m} q_j^{\alpha} - 1\right) A_1(x, p_1, p_2, \dots, p_m) + \frac{c_3}{n}\left(\sum_{i=1}^{m} p_i^{\alpha} - m\right). \end{aligned} \tag{17}$$

Fixing $P \in \Gamma_m^0$ such that $\sum_1^m p_i^\alpha - 1 \neq 0$ in (17), we obtain

$$A_1(x, q_1, q_2, \dots, q_m) = A_3(x)\left(\sum_{j=1}^m q_j^\alpha - 1\right) + a_3 \sum_{j=1}^m q_j^\alpha + b_3, \tag{18}$$

where a_3, b_3 are constants and A_3 is an additive function. As before, the additivity of $A_1(x, q_1, q_2, \dots, q_m)$ in x gives $a_3 \sum_1^m q_j^\alpha + b_3 = 0$. Thus (18) reduces to

$$A_1(x, q_1, q_2, \dots, q_m) = A_2(x)\left(\sum_{j=1}^m q_j^\alpha - 1\right), \tag{19}$$

which because of (16) gives $A_3(1) = 0$ so that $A_1(1, q_1, q_2, \dots, q_m) = 0$. Now we put (19) into (4) with $\alpha = \beta$ to obtain

$$\sum_{j=1}^m [f(pq_j) - p^\alpha f(q_j) - q_j^\alpha f(p)] = A_3(p)\left(\sum_{j=1}^m q_j^\alpha - 1\right). \tag{20}$$

We define

$$h(p) := f(p) + A_3(p), \ \ p \in]\,0,1\,[\tag{21}$$

By the use of (21) and the condition $A_3(1) = 0$, the equation (20) reduces to

$$\sum_{j=1}^m [h(pq_j) - p^\alpha h(q_j) - q_j^\alpha h(p)] = 0. \tag{22}$$

For temporarily fixed p, by using Result 1 on (22), we get

$$h(pq) - p^\alpha h(q) - q^\alpha h(p) = A_4(q - 1/m, p) \tag{23}$$

where $A_4 : \mathbf{R} \times]\,0,1\,[\to \mathbf{R}$ is an additive function in the first variable. For $p, q, r \in]\,0,1\,[$, by considering $h(pqr)$ first as $h(pq \cdot r)$ and then as $h(p \cdot qr)$ and using (23) we obtain

$$r^\alpha A_4(q - 1/m, p) + A_4(r - 1/m, pq) = p^\alpha A_4(r - 1/m, q) + A_4(rq - 1/m, p) \tag{24}$$

Putting $r = 1/m$ and using $A_4(0, x) = 0$ in (24), we get

$$(1 - m^{1-\alpha}) A_4(q, p) = (1 - m^{-\alpha}) A_4(1, p). \tag{25}$$

From (25), since A_4 is additive in the first variable, we get $A_4 = 0$ and then (23) reduces to

$$h(pq) = p^\alpha h(q) + q^\alpha h(p). \tag{26}$$

The general solution of (26) is given in Result 2. Thus from Result 2 and (21), we get

$$f(p) = A(p) + D(p)p^\alpha, \quad p \in]\,0,1\,[. \tag{27}$$

where $A(p)$ is an additive function and $D :]\,0,1\,[\to \mathbf{R}$ is a real valued function satisfying (3).

Remark. For the case $\alpha \neq \beta$, if one of the α or β is equal to 1, say for instance $\beta = 1$, we get the solution (14) proceeding exactly in the same manner. However, if $\alpha = 1$, then instead of fixing Q in Γ_m^0, we fix P in Γ_n^0 and proceeding in the same way we again get (14) as before.

Thus we have proved the following theorem.

THEOREM. *Let $f :]0,1[\to \mathbf{R}$ be a real valued function satisfying* (2) *for all $P \in \Gamma_n^0$ and $Q \in \Gamma_m^0$ with arbitrary but fixed $m, n (\geq 3)$ and $\alpha, \beta \notin \{0,1\}$. Then*

$$f(p) = \begin{cases} A(p) + d(p^\alpha - p^\beta, & \alpha \neq \beta \\ A(p) + D(p)p^\alpha, & \alpha = \beta \end{cases}$$

where $A(p)$ is an additive function with $A(1) = 0$ and d is an arbitrary, constant and $D :]0,1[\to \mathbf{R}$ is a real valued function satisfying (3).

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