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A NOTE ON THE TOPOLOGY ASSOCIATED WITH A LOCALLY CONVEX SPACE

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Abstract. We show that the barrelled (resp. σ-barrelled, d-barrelled) topology associated with a locally convex space (E,t) induces on a subspace F of countable conimension in E the associated barrelled (resp. σ-barrelled d-barrelled) topology. We also give a new proof of a few results from [8].

It has been shown in [1], [3], [4] and [12], that the properties of being barrelled, quasi-barrelled, bornological, σ-barrelled, d-barrelled, σ-quasi-barrelled, d-quasi-barrelled, b-barrelled, g-barrelled, p-space and b-space, are preserved under passage to subspaces of finite codimension. It is also known [6], [12], that a countable-codimesnional subspace of a barrelled (resp. σ-barrelled, d-barrelled) space is barrelled (resp. σ-barrelled, d-barrelled) space. On the other hand, the properties of being ultra-bornological, sequentially barrelled, k-barrelled and k-space are not preserved under passage to dense hyperplane [4], [5], [9], [11].

In general, if R is a property invariant under passage to an arbitrary inductive limit and the finest locally convex topology, then for every locally convex space (E,t) there exists a locally convex topology Rt, which is uniquely defined, i.e. $Rt = \lim\operatorname{ind} t_i$, where $t_i \geq t$ and t_i has the property R, for all $i \in I$ ([2], [8]). We say that Rt is the topology associated with a locally convex space (E,t). For example, R is one of the properties being barrelled, quasi-barrelled, ...

In this note we consider when the topology associated with a locally convex space (E,t) induces the topology associated with a subspace. We follow [7] and [8] for definitions concerning locally convex spaces. We shall need the following result of [8]:

If the linear mapping $f : (E,t) \to (F,p)$ is continuous, then $f : (E,Rt) \to (F,Rp)$ is continuous too.

We start with the following result:

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Theorem 1. *If R is a property invariant under projective topology, then from $(E,t) = \text{proj}\lim(E_i, f_i, t_i)$ it follows that $(E, Rt) = \text{proj}\lim(E_i, f_i, Rt_i)$, i.e. $R(\text{proj}\lim t_i) = \text{proj}\lim tRt_i$.*

Proof. By the result above it follows that $f : (E, t_i) \to (E, t)$ is again a continuous linear mapping from (E, Rt) in (E, Rt_i) for all i; according to the definition of projective topology we have that $Rt \geq \text{proj}\lim Rt_i$, and finally $Rt \leq \text{proj}\lim Rt_i$, since $\text{proj}\lim Rt_i$ has the property R. This completes the proof.

From this theorem we obtain:

Corollary 1. [8, Proposition I.8.1. and I.8.2]. *If F is a subspace of finite codimension in (E,t), then we have $Rt|F = R(t|F)$, where Rt is the barrelled (resp. σ-barrelled, d-barrelled, quasi-barrelled, σ-quasi-barrelled, d-quasi-barrelled, bornological) topology associated with the space (E, t_i) and $t|F$ is the relative topology on the subspace F.*

Corollary 2. *If F is a closed subspace of finite codimension in (E,t), then we have $Rt|F = R(t|F)$, where Rt is ultra-bornological (resp. k-barrelled, k-space) topology associated with the space (E,t).*

Corollary 3. *If F is a subspace of countable codimension in (E,t), then we have $Rt|F = R(t|F)$, where Rt is the barrelled (resp. σ-barrelled, d-barrelled topology associated with the space (E,t).*

Corollary 4. *If (E,t) is a topological product of a family (E_i, t_i), of locally convex spaces, then we have $Rt = \Pi Rt_i$, i.e. $R(\Pi t_i) = \Pi Rt_i$, where R, is a property invariant under topological product.*

Remark 1. We present here a direct and elementary proof that $Rt|F = R(t|F)$ where R is a property invariant under finite or countably codimensional subspace. The method used in [8] cannot be used to prove our Theorem and Corollary 3. Otherwise, the conclusion of Corollary 3 holds for every subspace with codimension less than c. We know that Valdivia has proved the following theorem: If (E,t) is a barrelled (resp. σ-barrelled, d-barrelled) space and F its subspace with codimension less than c, then $(F, t|F)$ is a barrelled (resp. σ-barrelled, d-barrelled).

From [2, Lemma 1.1] we know that if U is a barrel in a subspace F of finite codimension in a locally convex space (E,t), then there exists a barrel V in E such that $V \cap F = U$. From this it follows that the strong topology $\beta(E, E')$ induces on a subspace F the strong topology $\beta(F, F')$. If is a subspace of countable codimension, we have the following theorem:

Theorem 2. *Let (E,t) be a locally convex space such that $(E, \beta(E, E'))$ is a barrelled space and let F be a subspace of countable codimension in E; then the strong topology $\beta(E, E')$ induces the strong topology $\beta(F, F')$.*

Proof. Since $(E, \beta(E, E'))$ is a barrelled locally convex space, then the strong topology $\beta(E, E')$ is the barrelled topology associated with the space (E,t). Hence, according to Corollary 3 we have that $\beta(E, E')|F = Rt|F = R(t|F) \geq \beta(F, F')$. From [2], we know that $Rt \geq \beta(E, E')$ for every locally convex space (E,t), where R

is property of being barrelled. Otherwise, $\beta(F,F') \geq \beta(E,E')|F$, for every subspace F. Hence, $\beta(E,E')|F = \beta(F,F')$ and the proof of the theorem is completed.

COROLLARY. *If (E,t) is a locally convex space which satisfies the conditions of Theorem 2, then a subset A of E is strongly bounded in E, if and only if $A \cap F$ is strongly bounded in F.*

Remark 2. If F is a subspace of countable codimension in E, then examples A and B from [6] show that the strong topology $\beta(E,E)$ may not induce, the strong topology $\beta(F,F')$. The conclusion of Theorem 2 holds for every "subspace of codimension less than c. We do not know whether the condition $(E,\beta(E,E'))$ is a barrelled space" can be omitted from Theorem 2.

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