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A STUDY OF FULL COLLINEATION GROUP OF THE PROJECTIVE PLANE OF ORDER 26

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Abstract. It is proved that the full collineation group of a projective plane of order 26 is a $\{13, 5, 13, 23\}$-group.

1. Introduction. According to the Bruck-Ryser's theorem, the existence of a projective plane of order 26 is possible. This hypothetical projective plane we denote with $\mathbf{P}$, and with $G = operatornameAut\mathbf{P}$ we denote its full group of collineations.

We say that G is a π-group (π being a set of prime numbers) if π contains the set $\pi(G)$ of all prime divisors of $|G|$. By Cauchy's theorem, if the prime number p divides $|G|$, then G contains an element of order p. Therefore, the main task in building up the group G is to determine all possible collineations of prime order which operate on $\mathbf{P}$, and to find the set $\pi(G)$. In fact – by eliminating the possibility of collineations of $\mathbf{P}$ of many prime orders – we shall get a set π of prime numbers such that $\pi(G) \subseteq \pi$. By doing so we know that any collineation of the finite projective plane (see for example the classification in [3]) is either: (A) a quasiperspectivity (elation, homology, Baer's collineation), or (B) a generalized perspectivity, better to say semi-perspectivity (semi-elation, semi-homology), or (C) a planar collineation, or (D) a collineation which operates without fixed points. Let us mention that the type of a collineation α is determined by $\mathcal{F}(\alpha)$, here $\mathcal{F}(\alpha)$ denotes the fixed structurre of the collineation, i.e. the set of all elements (points and lines) of $\mathbf{P}$ which are fixed by α.

We shall prove the following

THEOREM. *The full collineation group of the projective plane of the order* 26 *is a* $\{3, 5, 13, 23\}$*-group.*

2. Some results. Almost all results below are contained in Hughes-Piper [1].

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The investigation of the existence of a finite projective plane or a subplane is always based on the following theorems:

Theorem 2.1. (Bruck-Ryser). *For* $n \equiv 1$ *or* $2 \pmod 4$ *there does not exist a projective plane of order* n *unless* n *is a sum of two integral squares.*

Theorem 2.2. (Bruck). *Let* $\mathbf{P}$ *be a projective plane of order* n *with a proper subplane* $\mathbf{P}_0$ *of order* m. *Then* $n = m^2$ *(Baer's subplane) or* $n \geq m^2 + m$.

For any collineation operating on a finite projective plane one has

Theorem 2.3. (Baer) *Every collineation of a finite projective plane has the same number of fixed points and fixed lines.*

Remark. 2.4. If Γ is a permutation group of the set $\mathcal{S}$, then, for any $x \in \mathcal{S}$, we denote by Γ_x the set of all elements of Γ which fix x. This Γ_x is called the stabiliser of x and it is easy to see that it is a subgroup of Γ. A special sort of permutation group is one in which Γ_x is always the identity; such a group is called semi-regular. (If a semi-regular group is also transitive, it is called regular). If the order of the collineation α is prime, then $\langle\alpha\rangle$ operates semi-regularly on $\mathbf{P} \setminus \mathcal{F}(\alpha)$. Moreover, $\langle\alpha\rangle$ operates semi-regularly on every α-invariant subset of $\mathbf{P} \setminus \mathcal{F}(\alpha)$. In this case the number of points of $\mathbf{P}(\alpha)$ (more precisely, the number of points of each α-invariant subset of $\mathbf{P} \setminus \mathcal{F}(\alpha)$), is divisible by p.

When considering the possibility of existence of an odd prime order collineation which operates with an even number of fixed points (lines) on a finite projective plane, the following theorem often leads to a contradiction (i.e. elimination).

Theorem 2.5. (Hughes). *Let* $\mathbf{P}$ *be a finite projective plane of order* n, *with a collineation a of prime order* $p > 2$. *Furthermore let the number of fixed points (lines) of* α *be even. Then the equation* $x^2 = ny^2 + (1)^{(p-1)/2}pz^2$ *has a nontrivial solution in* Z^3.

Theorem 2.6. *Every involution of a projective plane is a quasi-perspectivity.*

Theorem 2.7. (Hughes). *Let* $\mathbf{P}$ *be a projective plane of order* $n \equiv 2 \pmod 4$, *and suppose* P *has a collineation of even order. Then* $n = 2$.

3. Discussions about Hamyng's length and about "Spielprodukt". Supposse that the collineation α of prime order $p \geq 3$ operates on the projective plane P of order n. All nonfixed points from P are distributed on $\langle\alpha\rangle$-orbits of length p. These orbits have the from $\{I_0, I_1, \dots, I_{p-1}\}$ $(I = 1, 2, \dots, |P \setminus \mathcal{F}(\alpha)|/p$. where $|P \setminus \mathcal{F}(\alpha)|$ denotes the number of points of $P \setminus \mathcal{F}(\alpha)$. The point is determined by its orbital number I and by the index which denotes the ordinal number in the orbit. Let l be a line of P which does not contain any fixed point of α. By omitting the indices (but considering them as unknowns from $Zp = \{0, 1, \dots, p-1\}$), the line l has (so called orbital) representation

$$(*) \qquad l = \{\underbrace{1, 1, \dots, 1}_{s_1}, \underbrace{2, 2, \dots, 2}_{s_2}, \dots, \underbrace{m, m, \dots, m}_{s_m}\}.$$

It is allowed here that some multiplicities s_i be zero, but in each case $s_1+s_2+\ldots+s_m=n+1$. The number $H(l)=s_1(s_1-1)+s_2(s_2-1)+\cdots+s_m(s_m-1)$ is called Hamyng's length of l. With the assumption and notation given above, one has

Lemma 3.1. *Let α be a collineation of prime order p $(p\geq 3)$ of the projective plane P of order n and let a line l from P contain no fixed points of α. Then $H(l)=p-1$.*

Proof. Suppose that in the representation $(*)$ the points witn orbital number 1 have indices $x_1,x_2,\ldots,x_{s_1}$. Of course, these are different elements of Z_p; let $x_1<x_2<\cdots<x_{s_1}$. The line l intersects the line l^{α^i} in exactly one point for all $i=1,2,\ldots,p-1$. On the other hand, the line l will intersect the line l^{α^i} in a point with orbital number 1 for each

$$\begin{aligned} i\in\{&x_1-x_2,x_1-x_3,\ldots,x_1-x_{s_1},x_2-x_1,x_2-x_3,\ldots,x_2-x_{s_1},\ldots,x_{s_1}-x_1,\\ &x_{s_1}-x_2,\ldots,x_{s_1}-x_{S_1-1}\},\end{aligned}$$

where all these written differences are different $\mod p$ and none of them is zero mod p. According to this, there are exactly $s_1(s_1-1)$ such intersections. In the same way l intersects exactly $s_2(s_2-1)$ lines l^{α^i} in the points with orbital number 2, and so on. As the whole $\langle\alpha\rangle$-orbit of the line l has exactly p lines, we have exactly $p-1$ intersections of the line l with l^{α^i} $(i=1,2,\ldots,p-1)$, whence

$$p-1=s_1(s_1-1)+s_2(s_2-1)+\ldots+s_m(s_m-1),\ \text{ i.e. } H(l)=p-1.\quad\Box$$

Let us suppose that the lines l_1, and l_2 do not contain any fixed point of α and that l_1 and l_2 belong to different $\langle\alpha\rangle$-orbits. Let

$$l_1=\{\underbrace{1,1,\ldots,1}_{s_1},\ \underbrace{2,2,\ldots,2}_{s_2},\ldots,\ \underbrace{m,m,\ldots,m}_{s_m}\},$$
$$l_2=\{\underbrace{1,1,\ldots,1}_{t_1},\ \underbrace{2,2,\ldots,2}_{t_2},\ldots,\ \underbrace{m,m,\ldots,m}_{t_m}\};$$

$$s_1+s_2+\cdots+s_m=n+1,\ t_1+t_2+\cdots+t_m=n+1,\ H(l_1)=H(l_2)=p-1.$$

The number Spiel $(l_1,i_2)=s_1t_1+s_2t_2+\cdots+s_mt_m$ is called the “Spielprodukt” of the lines l_1 and l_2.

Lemma 3.2. *Spiel $(l_1,l_2)=p$.*

Proof. Of course, $|l_1\cap l_2^{\alpha^i}|=1$ for all $i=0,1,\ldots,p-1$. On the other hand, among the lines $l_2^{\alpha^i}$ $(i=0,1,\ldots,p-1)$ there are exactly s_1t_1 lines, which intersect l_1 in a point with orbital number 1 and so on $\ldots$, among the lines $l_2^{\alpha^i}$ $(i=0,1,\ldots,p-1)$ there are exactly s_mt_m lines which intersect l_1 in a point with orbital number m. Obvioussly, the sum total of these intersections must be p. Therefore, Spiel $(l_1,l_2)=p$. $\Box$

4. Proof of the Theorem. We shall now examine all possibilities of the type α, α being a collineation of prime order p acting on a projective plane P of order 26.

(A) α is an quasi-perspectivity

(A_1) α is an elation. The number of fixed points of α is $N = 27$ and by 2.4., $p \mid 26$, thus $p = 2$ or $p = 13$. Here $p = 2$ and $p = 13$ both divide $26^2 + 26 + 1 - 27 = 703 - 27 = 676$. According to 2.7., no involution ($n = 26 \equiv 2 \pmod 4$)) operates on P, so there only remains the possibility of the elation α of order 13.

(A_2) α is a homology. $p \mid 27 - 2 = 5^2$, and so $p = 5$. Furthermore $N = 28$ and $5 \mid 703 - 28 = 675$. Here is $N \equiv 0 \pmod 2$)), $p = 5 > 2$. The corresponding diophantine equation from 2.5. is $x^2 = 26y^2 + 5z^2$. By (2, Theorem 10.4.4.] we concude that this equation has no nontrivial solutions in the integers x, y, z. According to 2.5., there is no possibility for the homology α - a contradiction!

(B) α is a semiperspectivity.

(B_1) is a semielation. According to 2.6., α cannot be an involution. In view of 2.4., there only remains the possibility of a semielation a of prime order 13 which fixes $N = 1$ points (lines) of the plane P. In fact, α fixes an incidental pair (flag) point-line.

(B_2) α is a semihomology. There is a special possibility when the semihomology α of order 3 fixes a non-incidental pair point-line (thus $N = 1$). There remain other possibilities $N \in \{3, 8, 13, 18, 23\}$ whith $p = 5$. According to 2.5. the possibilities $N = 8$ and $N = 18$ are excluded, and i_t remains $N \in \{3, 13, 23\}$ with $p = 5$.

(C) a is planar.

According to 2.2., only subplanes of order 2, 3 and 4 are possible in a plane of order 26.

(C_1) The collineation α of prime order p fixes seven points (lines) of the subplane $P_0 = \mathcal{F}(\alpha)$ of order 2. From $p \mid 27 - 3 = 2^2 \cdot 3$ and $p \mid 703 - 7$ we conclude $p = 2$ or $p = 3$. By 2.6., it follows that $p = 3$.

(C_2) The collineation α fixes 13 points (lines) of the subplane $P_0 = \mathcal{F}(\alpha)$ of the order 3. In the same way as above we conclude $p = 23$.

Remark. The collineation α in this case fixes points ∞_i $(i = 0, 1, \ldots, 12)$ of the desarguesian subplane $P_0 = \mathcal{F}(\alpha)$ of order 3. The fixed lines of α observed in P are

$$\begin{aligned} &l_i = \{\infty_i, \infty_{i+1}, \infty_{4+i}, \infty_{6+i}, \;\; I_0, I_1, \ldots, I_{22}\} \\ &(i, I = 0, 1, \ldots, 12; \text{ all indices being taken } \mod 13). \end{aligned}$$

Thirteen $\langle\alpha\rangle$-orbits $\{I_0, I_1, \ldots, I_{22}\}$ (of length 23) of the points are distributed respectively on thirteen fixed lines l_i $(i = 0, 1, \ldots, 12)$ of α. There remain $703 - 13 - 23 \cdot 13 = 391$ points in P and they are distributed in 17 orbits $\{I_0, I_1, \ldots, I_{22}\}$ $(I = 13, 14, \ldots, 29)$. Dually, we have $13 \cdot 23 = 299$ lines, each containing exactly one point ∞_i (wich are distributed in 13 $\langle\alpha\rangle$-orbits of length 33) and $17 \cdot 23 = 391$ lines which do not contain any fixed point ∞_i (distributed in 17 $\langle\alpha\rangle$-orbits of length 23). Let r $(i = 13, 14, \ldots, 29)$ be lines belonging to the latter clas. The basic problem now is to give the orbital representation for r_i. Each r_i intersects

$l_i\,(i = 0, 1, \ldots, 12)$ in exactly one point. For this reason, in the representation of each r_i each of the orbital numbers $0, 1, \ldots, 12$ occurs exactly once. The remaining points (there are 14 of them) of each line r_i are represented by orbital numbers from $\{13, 14, \ldots, 29\}$. Let us mention that, according to 3.1. and 3.2., there must be $H(r_i) = 22$ $(i = 13, 14, \ldots, 29)$ and Spiel $(r_i, r_j) = p = 23$ for all $i \neq j$. For the representation of any of these lines, there exist, up to a renumeration, exactly four possibilities. So for the representation of r_{13} we have:

(Type a)

$$r_{12} = \{0, 1, \ldots, 12, 13, 13, 13, 13, 14, 14, 15, 15, 16, 16, 17, 17, 18, 18\}$$

or (Type b)

$$r_{13} = \{0, 1, \ldots, 12, 13, 13, 13, 13, 14, 14, 14, 15, 15, 16, 16, 17, 18, 19\}$$

or (Type c)

$$r_{13} = \{0, 1, \ldots, 12, 13, 13, 13, 14, 14, 14, 15, 15, 15, 16, 16, 17, 17, 18\}$$

or (Type d)

$$r_{13} = \{0, 1, \ldots, 12, 13, 13, 13, 13, 13, 14, 14, 15, 16, 17, 18, 19, 20, 21\}$$

Starting with r_{13} (one of the types given above) allows various possibilities for cotinuation with $r_{14}, r_{15}, \ldots$ – with regard to Hamyng's length and the "Spielprodukt". No matter how we start (the order will be arranged according to some choice of types) we come to the impossibility of the representiton at the fourth, fifth, or sixth step. So it is impossible to present r_{16} or r_{17} or r_{18}. In spite of a great number of tries we made, which always led to contradiction, this, of course, cannot be a proof. The problem is really at the very edge of "manual possibilities". But this leads us to a conjecture: No planar collineation of order 23 can act on the plane of order 26.

(C_2) The collineation α fixes 21 points (lines) of the subplane $P_0 = \mathcal{F}(\alpha)$ of order 4. We conclude, according to 2.4. and 2.6. that $p = 11$. Let us denote by ∞_i $(i = 0, 1, \ldots, 20)$ the fixed points of α. (These are the points of the desarguesian subplane $P_0 = \mathcal{F}(\alpha)$ of order 4.) Each fixed line $l_i\,(i = 0, 1, \ldots, 20)$ regarded as a line of the plane P, contains exactly $2\,\langle\alpha\rangle$-orbits (of lenghth 11). These are respectively:

$$\{I_0, I_1, \ldots, I_{10}\} \text{ and } \{I'_0, I'_0, \ldots, I'_{10}\}\ (I,\, I' = 0, 1, \ldots, 20).$$

So, we have

$$l_i = \{\infty_i, \infty_{i+1}, \infty_{6+i}, \infty_{8+i}, \infty_{18+i},\ \ I_0, I_1, \ldots, I_{10}, I'_0, I'_1, \ldots, I'_{10}\}$$

(the indices being taken modulo 21). There are exactly $(703 - 21)/11 = 62$ montrival $\langle\alpha\rangle$-orbits in the plane P. Of them, 42 are distributed on the fixed lines l_i (two on each). There remain exactly $20\,\langle\alpha\rangle$-orbits of points of P and let them be $\{I_0, I_1, \ldots, I_{10}$ $(I = 21, 22, \ldots, 40)$. So we have finally marked all points (703 of them) of P. In the plane P there are exactly $21 \cdot 11 \cdot 2 = 462$ lines containing exactly one fixed point of α. These lines are distributed in exactly $462/11 = 42$ $\langle\alpha\rangle$-orbits (two orbits for each ∞_i). There remain exactly $703 - 21 - 462 = 220$ lines which do not contain any fixed point of α. These lines should be distributed in exactly 20 $\langle\alpha\rangle$-orbits of length 11. The lines - representatives from these orbits - shall be denoted $r_i\,(i = 1, 2, \ldots, 20)$. Is the orbital representation for r_1, posible?

The line r_1 intersects each $l_i\,(i = 0, 1, \ldots, 20)$ in exactly one (non-fixed) point, and in its representation participate the orbital numbers $0, 1, \ldots, 20$ (each only once), and it is not essential whether these numbers are apostrophized or not. The Hamyng's number of each of these orbital numbers is $1(1-1) = 0$. For the remaining six points of the line $r_1\,(21 + 6 = 27)$ one has to use several orbital numbers from $\{21, 22, \ldots, 40\}$ with such multiplicities that - according to 3.1. – Hamyng's length of r_1 be $H(r_1) = p - 1 = 11 - 1 = 10$. But this is not possible. Therefore, there is no possibility for a planar collineation with the fixed subplane of the order 4 (in a plane of order 26).

(D) α operates without fixed points (i.e. $\mathcal{F}(\alpha) = \emptyset$). In this case $p \mid 703 = 19 \cdot 37$, thus $p = 19$ or $p = 37$. The corresponding equations from 2.5. are respectively:

$$x^2 = 26y^2 - 19z^2. \quad x^2 = 26y^2 + 37z^2.$$

Again by [2, Theorem 10.4.4.] we conclude that these equations have no nontrivial solutions in the integers x, y, z - a contradiction.

This finishes the proof that $\pi(G) \subseteq \{3, 5, 13, 23\}$, that is, the full group of collineations of a projective plane of order 26 is a $\{3, 5, 13, 23\}$-group. $\square$

Corollary. Each collineation (of prime order p) which operates on a projective plane of order 26 is of one of the following types:

1. an elation of order 13,
2. a semi-elaton of order 13 which fixes an incidental pair point-line,
3. a semi-homology of order 5 which fixes either 3 or 13 or 23 points or a semi-homology of order 3 which fixes one point,
4. a planar collineation of order 3, which fixes seven points and lines of a subplane of the order 2,
5. a planar collineation of order 23 which fixes thirtheen points of a subplane of order 3.

Finally we restate the very probable conjecture we made at the end of the Remark following the discussion of the case (C_2):

Conjecture. The full collineation group of a projective plane of order 26 is a $\{3, 5, 13\}$-group.

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