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ON CERTAIN CONDITIONS WHICH REDUCE A FINSLER SPACE OF SCALAR CURVATURE TO A RIEMANNIAN SPACE OF CONSTANT CURVATURE

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Abstract. We give certain conditions which reduce a Finsler space of scalar curvature to a Riemannian space of constant curvature.

1. Preliminaries. Let F_n be an n-dimensional Finsler space with the fundamental functional $L(x,y)$, the positive definite metric tensor $g_{ij} = 1/2\dot{\delta}_i\dot{\delta}_j L^2$ and the angular metric tensor $h_{ij} = g_{ij} - l_i l_j$, where $l_i = \dot{\delta}_i L$, $\dot{\delta}_i = \delta/\delta y^i$.

For a Cartan connection $C\Gamma$, h-and ν-covariant derivatives of a finsler tensor field X^i_j are denoted by $X^i_{j|k}$ and $X^i_j|_k$. The h-, $h\nu$- and ν-curvature tensors of $C\Gamma$ are R^i_{hjk}, P^i_{hjk} and S^i_{hjk} and the (ν) h-, (h) $h\nu$- and (ν) $h\nu$-torsion tensors of $C\Gamma$ are R^i_{jk}, C^i_{jk} and P^i_{jk} respectively. On the otherhand H^i_{hjk} and H^i_{jk} are h-curvature tensors and (ν) h- torsion, tensors of Berwald connection $B\Gamma$ respectively.

The following relations are well known [4]:

$$H^h_{ijk} = \dot{\delta}_i H^h_{jk} \tag{1.1}$$

$$P_{ijk} = C_{ijk|o}, \tag{1.2}$$

where the index o stands for transvection by y and $C_{ijk} = 1/2\dot{\delta}_k g_{ij}$

$$H^i_{jk} = H^i_{ojk} = R^i_{jk} = R^i_{ojk}, \tag{1.3}$$

$$H^h_{ijk} = R^h_{ijk} - C^h_{ir}R^r_{jk} + \{P^h_{ij|k} - P^h_{jr}P^r_{ki} - j \mid k\}. \tag{1.4}$$

where $j \mid k$ means interchange of indices j, k in the foregoing terms.

A hypersurface of F_n defined by the equation $L(x,y) = 1$, where the point $x = (x^i)$ is fixed and y^i are variables, is called indicatrix. We denote by $p\cdot$ the projection of the tensor of the Finsler spaces on the indicatrix. For example, the

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projection of the tensor T^i_j of type (1,1) of F_n on the indicatrix is $p\cdot T^i_j = h^i_a T^a_b h^b_j$, where $h^i_a = \delta^i_a - l^i l_a$, $l^i = g^{ij} l_j = L^{-1} y^i$. A tensor T satisfying $p\cdot T = T$ is called an indicatric tensor. We have

$$\text{(1.5)}\qquad \begin{aligned} &\text{a) } p\cdot l^i = p\cdot l_i = 0, && \text{b) } p\cdot \delta^i_j = h^i_j,\\ &\text{c) } p\cdot \dot\delta_k h^i_j = p\cdot h^i_j|_k = 0, && \text{d) } p\cdot \dot\delta_k h_{ij} = 2C_{ijk} \end{aligned}$$

2. A Finsler space of scalar curvature. A Finsler space of scalar curvature is characterized by [6] any one of the following equations:

$$\text{(2.1a)}\qquad H^i_{jk} = L(Kl_j + K_j/3)h^i_k - j \mid k,$$

$$\text{(2.1b)}\qquad \begin{aligned} H^i_{hjk} = [\{&l_h(Kl_j + K_j/3) + Kh_{hj} + 2K_h l_j/3 + K_{hj}/3\}h^i_k\\ &+ l^i(Kl_k + K_k/3)h_{hj} + h^i_h l_j K_k/3] - j \mid k \end{aligned}$$

where $K_k = L\dot\delta_k K$, $K_{hj} = Lp\cdot \dot\delta_h K_j = K_{jh}$. Specially, if the scalar K is constant, then the space is called a Finsler space of constant curvature.

PROPOSITION 2.1. *A Finsler space $F_n (n \geq 3)$ of scalar curvature K satisfies*

$$\text{(2.2)}\qquad K_{ijk} + K_i h_{jk} - i \mid j = 0,$$

where $K_{ijk} = Lp\cdot \dot\delta_i K_{jk}$.

Proof. From (1.5) and (2.1b), we have

$$\begin{aligned} Lp\cdot \dot\delta_m H^i_{hjk} = (&h_{hm}K_j/3 + K_m h_{hj} + 2LKC_{hjm} + 2K_h h_{jm}/3 + K_{mhj}/3)h^i_k\\ &+ h^i_m K_k h_{hj}/3 + h^i_h h_{jm} K_k/3 - j \mid k \end{aligned}$$

Considering the skew-symmetric parp of the above equation in the indices h and m and using the fact $\dot\delta_m H^i_{hjk} = \dot\delta_h H^i_{mjk}$, we get

$$[(K_m h_{hj} + 2K_h h_{jm}/3 + K_{mhj}/3)h^i_k - j \mid k] - h \mid m = 0$$

which is simplified as

$$\text{(2.3)}\qquad [(K_m h_{hj} + K_{mhj})h^i_k - j \mid k] - h \mid m = 0$$

Contracting (2.3) in indices i and k, we get (2.2).

Remark 2.1. Proposition 2.1. and the definition of K_j, K_{hj} and K_{ijk} imply that when any one of them is zero, then the other two are automatically zero. $K_j = 0$ means that K is independent of y. Thus K is constant (Matsumoto [4, Prop. 26.1]). If for a Finsler space F_n of scalar curvature any one of the tensors K_i, K_{hj} and K_{ijk} vanishes, F_n is of constant curvature.

PROPOSITION 2.2. *A Finsler space F_n of scalar curvature K with $P_{hij|0} = 0$ satisfies*

$$\text{(2.4)}\qquad h_{ih}(3KK_{jm} - K_j K_m) + h_{jh}(3KK_{im} - K_i K_m) - h \mid m = 0$$

Proof. A Finsler space F_n of scalar curvature K satisfies [7]

$$L^{-1}P_{hij|0} + LKC_{hij} + (K_h h_{ij} + K_i h_{hj} + K_j h_{hi})/3 = 0. \tag{2.5}$$

Since, $P_{hij|0} = 0$, (2.5) leads to

$$LKC_{hij} + (K_h h_{ij} + K_i h_{hj} + K_j h_{hi})/3 = 0. \tag{2.6}$$

Differentiating the equation above partially with respect to y^m and applying p to the resulting equation and using (1.5) we get

$$3LK_m C_{hij} + 3L^2 Kp \cdot \dot{\delta}_m C_{hij} + (2LC_{ijm}K_h + h_{ij}K_{hm} + 2LC_{jhm}K_i + h_{jh}K_{im} + 2C_{him}K_j + h_{hi}K_{jm}) = 0.$$

Considering skew symmetric part of the above equation in the indices h and m, we get

$$LC_{hij}K_m + h_{jh}K_{im} + h_{hi}K_{jm} - h \mid m = 0. \tag{2.7}$$

By virtue of (2.6) and (2.7), we obtain (2.4).

A Riemannian space is characterized by [4]:

$$C_{hij} = 0. \tag{2.8}$$

THEOREM 2.3. *A Finsler space F_n of non-vanishing scalar curvature K with $P_{hij|0} = 0$ is a Riemannian space of constant curvature if*

$$3KK^m_m - K^m K_m = 0, \tag{2.9}$$

where $K^i_j = g^{im}K_{mj}$, $K^i = g^{im}K_m$.

Proof. Transvecting (2.4) by $h^{ih} = g^{ih} - l^i l^h$ we get

$$(n-1)(3KK_{jm} - K_j K_m) - (3KK^s_s - K^s K_s)h_{jm} = 0$$

which leads to

$$3KK_{jm} - K_j K_m = 0 \tag{2.10}$$

because of (2.9).

Differentiating (2.10) partially with respect to y^h and applying p to the resulting equation, we have

$$3K_h K_{jm} + 3KK_{hjm} - K_{hj}K_m - K_j K_{hm} = 0 \tag{2.11}$$

Equations (2.10) and (2.11) give $K_m K_h K_j + 9K^2 K_{hjm} = 0$ which yields

$$K_{hjm} - h \mid j = 0 \quad K \neq 0, \tag{2.12}$$

By virtue of (2.2) and (2.12), we get $K_h h_{jm} - h \mid j = 0$ which shows that

$$K_h = 0. \tag{2.13}$$

On account of remark 2.1 and equations (2.6), (2.8) and (2.13), we have the theorem.

COROLLARY 2.4. *A Finsler space F_n of non-vanishing constant curvature ($K_j = 0$, $K \neq 0$) with $P_{hij|0} = 0$ is a Riemannian space of constant curvature.*

Proof. Since F_n is of constant curvature, we get $K_j = K_{hj} = 0$. Thus all the conditions of theorem 2.3 are fulfilled. Hence the corollary.

The h-curvature tensor of Rund connection is defined as follows [4]:

$$K^i_{hjk} = R^i_{hjk} - C^i_{hr} R^r_{jk}. \tag{2.14}$$

THEOREM 2.5. *A Finsler space $F_n (n \geq 3)$ of non-vanishing scalar curvature K is a Riemannian space of constant curvature if the h-curvature tensor of Berwald and Rund coincide.*

Proof. From (1.4) and (2.14), we obtain $P^h_{ij|k} - P^h_{jr} P^r_{ki} - j \mid k = 0$ which implies $P_{ihj|k} - P_{jhr} P^r_{ki} - j \mid k = 0$. Considering symmetric part of the above equation in i and h, we have

$$P_{ihj|k} - \supset| \; k = 0 \tag{2.15}$$

Also from (1.4), we get

$$H_{ihjk} + H_{hijk} = -2C_{ihr} R^r_{jk} + 2(P_{ihj|} - j \mid k) \tag{2.16}$$

Substitution of (2.15) in (2.16) gives

$$H_{ihjk} + H_{hijk} = -2C_{hir} R^r_{jk} \tag{2.17}$$

By virtue of (2.1a) and (2.1b), we obtain

$$p \cdot H^i_{jk} = LK_j h^i_k / 3 - j \mid k \tag{2.18a}$$

$$p \cdot H_{hijk} = (Kh_{hj} + K_{hj}/3) h_{ik} - j \mid k \tag{2.18b}$$

Applying indicatric projection $p \cdot$ on (2.17) and using (2.18a) and (2.18b) we get

$$K_{ij} h_{hk} + K_{hj} h_{ik} - j \mid k = -2LK_j C_{hik} - j \mid k. \tag{2.19}$$

Since $P_{ihj|0} = 0$ because of (2.15), using (2.5) and (2.19), we have

$$(3KK_{ij} - 2K_i K_j) h_{hk} + (3KK_{hj} - 2K_h K_j) h_{ik} - j \mid k = 0 \tag{2.20}$$

(2.4) and (2.20) lead to $K_i K_j h_{hk} + K_h K_j h_{ik} - j \mid k = 0$. Transvecting the last relation by h^{hk}, we get

$$(n-1) K_i K_j - K^m K_m h_{ij} = 0 \tag{2.21}$$

Transvecting the relation above by K^iK^j, we obtain $(n-2)K^mK_mK^sK_s = 0$, which implies $K^mK_m = 0$. Hence $K_i = 0$ identically. By definition $K_{hj} = 0$ also. Thus all the conditions of theorem 2.3 are satisfied. Hence the Theorem.

T-tensor T_{hijk} is defined by [3]

$$T_{hijk} = LC_{hij}\big|_k + l_hC_{ijk} + l_iC_{hjk} + l_jC_{hik} + l_kC_{hij}. \tag{2.22}$$

Ikeda [2] has proved that Finsler tensor field u^i sastisfies

$$u^i_{j|0}\big|_k - u^i_{j|k} - u^i_j\big|_{k|0} = 0 \tag{2.23}$$

THEOREM 2.6. *A Finsler space of non-vanishing scalar curvature with vanishing T-tensor and $P_{hij|0} = 0$ is a Riemannian space of constant curvature if $C_{hij|k|0} = 0$.*

Proof. Since $T = 0$, (2.2) implies

$$LC_{hij|k} = -l_hC_{ijk} - l_iC_{hjk} - l_jC_{hik} - l_kC_{hih}. \tag{2.24}$$

Differentiating (2.24) h-covariantly and transvecting the resulting equation by y and using (2.23), we obtain

$$L(P_{hii}\big|_k - C_{hij|k}) = l_hP_{ijk} - l_iP_{hjk} - l_jP_{hik} - l_kP_{hij}$$

Differentiating the equation above h-covariantly once again and transvecting the resulting equation by y and using (2.23), $P_{hij|0} = 0$ and $C_{hij|h|0} = 0$, we obtain

$$P_{hij|k} = 0. \tag{2.25}$$

From (2.15) – (2.21) and (2.25) we have $K_i = 0$ and consequently $K_{hj} = 0$.

Thus the theorem follows in light of Theorem 2.3.

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