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SOME ANALYTIC METHODS WITH APPLICATIONS TO NUMBER THEORY

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Abstract. We study the arithmetical function "exponent (order) of an integer modulo m" which is here shortly named "period" of m. A method is developed, named "separation of parameters", that leads to analytic representation of the function period. Though Bessel functions have dominant role, other special functions are also applicable. The most promising result is derived by making use of Mukisiński's concept of distributions. The developed method, besides its general nature, makes it possible to study computability of arithmetical function period by means of analytic procedures.

1. Introduction

Let a and m be relatively prime integers, $m > 1$. The smallest positive integer k such that $a^k \equiv 1(\mod m)$ is called exponent (order) of a modulo m, and is denoted by $k = \exp_m(a)$. For the sake of simplicity we shall denote this function by $k(a,m)$, and call it period. Throughout the following discussion a is fixed, and for the present the most important particular case is $a = 2$. We shall show that it is possible to construct analytic interpolation of the function $k(a,m)$. The method which will yield such analytic expression is named here "separation of parameters". Similar method, just slightly modified, can be applied to several problems of number theory, such as Diophantine equations etc. The general procedure may be summarized as follows. Suppose that we have an expansion

$$\exp(ixy) = \sum_{n=0}^{\infty} f_n(x) g_n(y), \tag{1}$$

and let a properly chosen periodic distribution $\varphi(x)$ with the period $2\pi/m$ is expanded in its Fourir series

$$\varphi(x) = \sum_{j=-\infty}^{\infty} \alpha_j \exp(-jmxi). \tag{2}$$

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Next, assume that after multiplying (1) by $\varphi(x)\exp(-ix)$ the integration of series may be done term by term:

$$\int_0^{2\pi} \exp(ix(y-1))\varphi(x)dx = \sum_{n=o}^{\infty} a_n g_n(y), \tag{3}$$

where

$$a_n = \int_0^{2\pi} f_n(x)\varphi(x)\exp(-ix)dx. \tag{4}$$

It should be noted that we use distributions in the sense of Mikusiński, hence the integral of distributions is in his sense, cf. [**1**]. Also, it is clear that an depends on m. Put now $y = a^s$, $s \in \mathbf{N}$, in (3) and assume that series (2) can be substituted in (3) and integrated term by term. If we use notation

$$h(s) = a^s - 1 \tag{5}$$

we obtain

$$\sum_{n=0}^{\infty} a_n g_n(a^s) = \int_0^{2\pi} \exp(ixh(s))\varphi(x)dx = \begin{cases} 2\pi\alpha_j & \text{if} \quad j = h(s)/m \in \mathbf{Z} \\ 0 & \text{if} \quad m \nmid h(s). \end{cases} \tag{6}$$

It is well-known that $h(s)$ is divisible by m if and only if $s = uk$, $u \in \mathbf{N}$. In order to obtain sufficiently fast convergence, multiply (6) by a weighting function $\gamma(s)$ and take summation over $s \in \mathbf{N}$. Particular cases developed in this paper show that such a function γ can be found so that the resulting double sum admits change of order of summation. Performing the described procedure we get

$$\sum_{n=0}^{\infty} a_n c_n = 2\pi \sum_{u=1}^{\infty} \gamma(uk)\alpha_{h(uk)/m} \tag{7}$$

where

$$cn = \sum_{s=1}^{\infty} \gamma(s) g_n(a^s). \tag{8}$$

Now let us emphasize the following facts. In (7), disregarding obvious dependence on indices, a_n depends on m, c_n depends on a, whereas the right-hand side depends on a, m and k. Therefore we can state

THEOREM 1. *Concerning the arithmetical function $k(a,m)$, $(a,m) = 1$, the relation* (7) *together with* (2), (4) *and* (8) *gives an implicit connection among the parameters a, m and k.*

Investigations of numerous particular examples showed that the most important role in this method has the choice the distribution $\varphi(x)$. Roughly speaking, the method is more fruitful when the proper distribution $\varphi(x)$ resembles the ordinary function less.

Behavior of the function $k(a,m)$ for $a = 2$ and large m is closely related to the question of cardinal numbers of Mersenne and Fermat primes. We should remind that primes of the form $2^p - 1$ are called Mersenne, and primes of the form $2^{2^t} + 1$ ar called Fermat. The method of separation of parameters has been originally developed in order to derive possible consequences from the following two propositions.

PROPOSITION 2. *If number of Mersenne primes is infinite, then $k(2,m)/\log_2 m$ approaches 1 (over numbers greater than 1) as m tends to infinity over Mersenne primes.*

PROPOSITION 3. *If number of Fermat primes is infinite, then $k(2,m)/\log_2 m$ approaches 2 (over numbers less than 2) as m tends to infinity over Fermat primes.*

We also mention that obtained analytic expressions for $k(a,m)$ comprise functions suitable for further applications of probabilistic methods.

2. Application of Bessel functions

The number of diverse expansions (1) is infinite. It is not easy to state general conditions which must be imposed on the functions f_n and g_n in order that the method of separation of parameters will work. Also, particular cases suggest that there are present unavoidable limitations which make impossible any improvement of the method by the choice of the expansion (1). The two following expansions were tested, and may serve as starting point for the method described in the preceding section. These are:

a) Jacobi expansion, cf. [p. 22, 8]

$$\exp(ixy) = \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(y) T_n(x), \quad -1 \le x \le 1,$$

where ε_n is Neumann's factor ($varepsilon = 1$, else $\varepsilon = 2$), $T_n(x)$ denotes Tchebichef polynomials;

b) Generating formula for Laguerre polynomials, cf. [p. 189, (17), 2],

$$\exp(ixy) = \sum_{n=0}^{\infty} i^{-n} y^n (1 - iy)^{-n-1/2} L_n^{-1/2}(x).$$

Probably the best results could be obtained by means of the degenerate form of Gegenbauer's addition theorem, cf. [p. 368, 8], or [p. 213, 2]; it is sometimes called Sonin's formula, cf. [p. 64, 2]:

$$\exp(ixy) = 2^{\lambda}\Gamma(\lambda) \sum_{n=0}^{\infty} i^n (n+\lambda) y^{-\lambda} J_{n+\lambda}(y) C_n^{\lambda}(x), \tag{9}$$

$-1 \le x \le 1$. Though formula (9) is applicable for $1/2 \le \lambda \le 3/2$, we shall consider the particular case $\lambda = 1/2$, when Gegenbauer's polynomials reduce to Legendre polynomials: $C_n^{1/2}(x) = P_n(x)$. Next, take in (9) $y = a^s\pi$, $s \in \mathbf{N}$, and replace x by $\pi^{-1}x - 1$ to obtain

$$\exp(ixa^s) = (-1)^a 2^{1/2} \sum_{n=0}^{\infty} b_n P_n(\pi^{-1}x - 1), \tag{10}$$

where

$$b_n =^n (n + 1/2)a^{-s/2} J_{n+1/2}(a^s\pi), \tag{11}$$

$0 \leq x \leq 2\pi$. The series in (10) is uniformly convergent with regard to x in $[0, 2\pi]$.

We shall separately consider the two cases: *Case* 2.1. φ is a classical bounded function: $|\varphi(x)| \leq M$, $x \in \mathbf{R}$. *Case* 2.2. φ is constructed by means of δ-distribution.

Case 2.l. Relatively simple results can be obtained if one takes for $\varphi(x)$ periodic *Bernoulli* or *Euler polynomials*:

$$\left(\frac{2\pi}{m}\right)^r \frac{1}{r!}\overline{B}_r((mx/2\pi) + \beta) \quad \text{or} \quad \left(\frac{\pi}{m}\right)^r \frac{1}{r1}\overline{E}_r((mx/\pi) + \beta),$$

where $r \in \mathbf{N}$, $\beta \in [0, 1)$; note that $\overline{B}_r(x)$ resp. $\overline{E}_r(x)$ coincide with Bernoulli resp. Euler polynomials for $x \in (0, 1)$. However, the best results are obtained by the following choice:

$$\varphi(x) = \sum_{s=0}^{\infty} \binom{-v-1}{s} \psi_{\nu+s}(x)\psi_\nu(x) = \frac{1}{2i^{\nu+1}} \frac{\pi^{\nu+11}}{\nu! m^\nu} \overline{E}_\nu(mx/\pi), \tag{12}$$

$\nu = 0, 1, 2, \ldots$ From $|\overline{E}_\nu(y)| \leq \nu!\pi^{1-\nu}/2$, $y \in (0, 2)$, we find that $|\varphi(x) \leq \pi^2 m(m-1)^{-\nu-1/4} = M$ for $x \in (0, 2\pi)$. The series (12) is uniformly convergent with regard to x in $(0, 2\pi)$. Calculation of the Fourier expansion of the function $\varphi(x)$ is facilitated by similar expansion of Euler polynomials, cf. [p. 66, 6]. The result reads:

$$\varphi(x) = m {\sum_{j \in \mathbf{Z}}}' (mj + 1)^{-\nu-1} \exp(-jmxi), \tag{13}$$

where $m > 1$, $\nu \geq 0$ and $\sum'_{j \in \mathbf{Z}}$ denotes (through entire text) summation over all odd integers. Multiplying (10) by $\varphi(x)\exp(-ix)$ we obtain the series which is also uniformly convergent with regard to x in $(0, 2\pi)$, and therefore it can be integrated term by term. Thus we get

$$\int_0^{2\pi} \varphi(x) \exp(ih(s)x)dx = (-1)^a 2^{1/2} \sum_{n=0}^{\infty} a_n b_n, \tag{14}$$

where

$$a_n = \int_0^{2\pi} P_n(\pi^{-1}x - 1)\varphi(x)\exp(-ix)dx. \tag{15}$$

Hence $|a_n| \leq 2\pi M$. Since $P_n(\pi^{-1}x - 1)\exp(-ix)$ and $\varphi(x)$ are square-summable functions in $(0, 2\pi)$, we can apply a known theorem (cf. [p. 575, 4]), to obtain

$$a_n = m {\sum_{j \in \mathbf{Z}}}' (mj + 1)^{-\nu-1} \int_0^{2\pi} P_n(\pi^{-1}x - 1)\exp(-ix(mj + 1))dx.$$

The integral is already calculated as Fourier exponential transform, cf. [p. 122, (1), 3]. Thus we have

$$a_n = (-1)^{m+1} i^{-1} m\pi 2^{1/2} {\sum_{j\in\mathbf{Z}}}' (mj+1)^{-\nu-3/2} J_{n+1/2}(\pi(mj+1)), \tag{16}$$

where $m = 2, 3, 4, \ldots,\ \ n = 0, 1, 2, \ldots,\ \ \nu = 0, 1, 2, \ldots.$ It is well-known that Bessel functions whose order is half of an odd integer are expressible in finite terms by means of elementary functions (cf. for instance [p. 53, 8] or [p. 78, 2]). This fact leads to the following method of transformation of sequences. In the first place we introduce the sequence of functions

$$\begin{aligned} A_{2r+1}(z) &= \sum_{s=0}^{r} (-1)^s \frac{(2r+2s+1)!}{(2s)!(2r-2s+1)!} (2z)^{-2s} \\ B_{2r}(z) &= \sum_{s=0}^{r-1} (-s)^s \frac{(2r+2s+1)!}{(2s+1)!(2r-2s-1)!} (2z)^{-2s-1} \end{aligned} \tag{17}$$

$r = 0, 1, 2, \ldots$, generated by functions $J_{n+1/2}(z)$. Simpler notation is actived by

$$C_n(z) = \begin{cases} A_n(z) & \text{for} \quad \text{odd } n \\ B_n(z) & \text{for} \quad \text{even } n. \end{cases}$$

Next, given a sequence $t_{-n},\ n = 0, 1, 2, \ldots$, we adopt the usual symbolic power notation putting $t^{-n} = t_{-n}$. In this way relations (17) can be understood as transformation of sequence $(t_{-n})_n$ into the sequence $C_n(t),\ n = 0, 1, 2, \ldots.$ Now we can derive more convenient expression for coefficients an given by (16). Let us introduce the two-parameter sequence

$$t_{-n} = {\sum_{j\in\mathbf{z}}}' (mj+1)^{-n-\nu-2}, \tag{18}$$

$n = 0, 1, 2, \ldots,\ m = 2, 3, 4, \ldots,\ \nu = 0, 1, 2, \ldots.$ The absolute convergence of series makes it possible to bring (16) in the following (symbolic) form

$$a_n = (-1)^{n(n-1)/2} i^n 2m C_n(\pi t). \tag{19}$$

For the weighting function we shall take $\gamma(s) = a^{-\rho s},\ \rho \in \mathbf{R}$. Further, we observe that $\varphi(x)$ and $\exp(ixh(s))$ are square-summable functions, hence (6) is valid in this case. Now relations (14), (5), (6) and (13) yield

$$2\pi m(a^{k(\rho+\nu+1)} - 1)^{-1} = (-1)^a 2^{1/2} \sum_{s=1}^{\infty} a^{-\rho s} \sum_{n=0}^{\infty} a_n b_n. \tag{20}$$

At this point we need a property of Bessel functions.

Lema 4. *The asymptotic formula*

$$\sum_{n=0}^{\infty} |J_{n+1/2}(z)| = O(z^{3/2}) \tag{21}$$

holds for large z.

Proof. If we put $\mu = \nu = n + 1/2$ in [p. 150, (1), 8], it reduces to

$$d_n = J^2_{n+1/2}(z) = \frac{2}{\pi}\int_0^{\pi/2} J_{2n+1}(w)dx$$

where $n = 0, 1, 2, \ldots,\ w = 2z\cos x$ and we assume that z is positive real variable. By the threefold application of the recurrence formula

$$J_\nu(w) = w(J_{\nu-1}(w) + J_{\nu+1}(w))/(2\nu) \tag{22}$$

[p. 45, (1), 8] it follows that

$$\begin{aligned} d_n =& \frac{z^3}{2(2n+1)}\int_0^{\pi/2}\left\{\frac{1}{n(2n-1)}J_{2n-2}(w) + \frac{3}{(n+1)(2n-1)}J_{2n}(w) + \right.\\ &\left. + \frac{3}{n(2n+3)}J_{2n+2}(w) + \frac{1}{(n+1)(2n+3)}J_{2n+4}(w)\right\}\cos^3 x dx, \end{aligned}$$

$n = 1, 2, \ldots$. Taking into account the Lommel inequality $|J_\nu(w)| \le 1$ for $\nu \ge 0$, $w \in \mathbf{R}$, [p. 406, (10), 8], we obtain

$$d_n < \frac{z^3}{(2n-1)(2n+1)(2n+3)}\int_0^{\pi/2}\cos^3 x dx.$$

whence $d_n^{1/2} = z^{3/2}\,O(n^{-3/2})$, $n \ge 1$, uniformly with regard to $z \in (0, \infty)$. Therefore

$$\sum_{n=0}^{\infty} d_n^{1/2} = |J_{1/2}(z)| + z^{3/2}\sum_{n=1}^{\infty} O(z^{-3/2}),$$

which leads to the assertion (21).

The following corollary to Lemma 4 is essential for the development of the method.

Corollary 5. *Let r be nonnegative integer, α and β positive reals. Then for large z*

$$\sum_{n=0}^{\infty}(\alpha n + \beta)^r |J_{n+1/2}(z)| = O(z^{r+3/2}). \tag{23}$$

Proof is by induction on r, the base is given by (21) and steps are derived by (22).

We now proceed with changing the order of summation in (20). One sufficient condition for that is as follows (cf. for instance [p. 52, 4]). We shall show that inner series in (20) converges absolutely, and after substitution of that series of absolute values by convenient upper bound, the remaining series also converges absolutely. Then the double series in (20) is absolutely convergent, and change of order of summation is justified. From (16), with $\nu > -1/2$ and fixed in, it is easily seen that czn is absolutely bounded. Now, by Corollary 5, with $r = 1$, $\alpha = 1$, $\beta = 1/2$, the inner sum in (20) is estimated to be $O(a^{2s})$. Hence the double sum in (20) converges absolutely if $rho > 2$. Performing the described procedure, (20) is equivalent to

$$2\pi m(a^{k(\rho+\nu+1} - 1)^{-1} = \sum_{n=0}^{\infty} i^n (n+1/2) a_n c_n \tag{24}$$

where

$$c_n = (-1)^a 2^{1/2} \sum_{s=1}^{\infty} a^{-(\rho+1/2)s} J_{n+1/2}(a^s \pi). \tag{25}$$

The transformation that led from (16) to (19) now may be applied again. From this reason introduce the two-parameter sequence

$$\nu_{-n} = (a^{n+\rho+1} - 1)^{-1}, \tag{26}$$

$n = 0, 1, 2, \ldots$, with the property of symbolic power: $\nu^{-n} = \nu_{-n}$. Then (25) yields

$$c_n = -(-1)^{n(n+1)/2} 2\pi^{-1} C_n(\pi\nu). \tag{27}$$

Finally, (24) by means of (19) and (27) can be brought into the form

$$(a^{k(\rho+nu+1)} - 1)^{-1} = 2\pi^{-2} \sum_{n=0}^{\infty} (n+1/2) C_n(\pi t) C_n(\pi\nu), \tag{28}$$

from which k is explicitly expressible by elementary functions.

The series in (28) is alternating, with absolute terms not too small. However, the left-hand side in (28) shows that the sum of the series can be very small, if a, ρ, ν and k are large. Hence, concerning numerical applications of (28), there is present the so called loss of accuracy. This effect can be reduced by choosing o and v as small as possible. Numerical evidence shows that exact estimate of the sum in (21) could be $(z/e)^{1/2}$. In that case the choice of smaller ρ, namely $\rho > 1$, would be possible. The choice of smaller ν becomes possible, if for φ in (2) we put a proper distribution. That is only one of goals which we have in mind developing Case 2.2.

The applied transformation of sequences in (19) and (27) is essentially based on the fact that a and m were integers. Considering the question of interpolation of the function period, and even a possibility of analytic continuation of this function, we must observe that $(-1)^m a_n$ expressed by (16) has meaning for every real $m > 1$ and every integer $\nu \geq 0$. Also under these conditions $(-1)^m a_n$ is still bounded with respect to n (proof by employing Lommel's inequality). However, a_n given by (19) has meaning for every integer $\nu > 0$ and every complex m distinct from the reciprocal of an odd integer. Expressions (16) and (19) have equal values for integers $m > 1$, but need not be equal functions of real m. Examination of the properties of the coefficients a_n given by (19) and considered as functions of yn in the complex plane seems to be a very difficult problem. We do not know whether these functions are bounded with respect to n, even if m runs only over reals,

$m > 1$. Coefficients $(-1)^a c_n$ given by (25) retain meaning if a ranges over reals greater than 1. In that case series in (24) remains convergent. Thus we have further possibility for interpolation of the function period.

In the most important case $a = 2$, the integer m must be odd, hence factor $(-1)^{m+1}$ in (16) may be dropped. Thus the obtained expression for a_n, interpolated as the function of real variable m, we denote by $(-1)^{n(n-1)/2} i^n 2m a_n^*(m)$, whence

$$a_n^*(m) = (-1)^{n(n+1)/2} \pi 2^{-1/2} \sum_{j \in \mathbf{Z}}^{\infty} (mj+1)^{-\nu-3/2} J_{n+1/2}(\pi(mj+1)) \tag{29}$$

where $n = 0, 1, 2, \ldots,\ \nu = 0, 1, 2, \ldots,\ m$ real number greater than 1. We know that $a_n^*(m),\ (n = 0, 1, 2, \ldots)$ are continuous functions of m so that for fixed ν and ρ the function

$$D^*(m, 2; \nu, \rho) = 2\pi^{-2} \sum_{n=0}^{\infty} (n + 1/2) a_n^*(m) C_n(\pi\nu)$$

is also continuous for real $m > 1$. Denote by $S_{\nu,\rho}$, the set of zeros of this function. From (6) one concludes that all positive even integers belong to $S_{\nu,\rho}$. Computer calculations show that all remaining zeros are located closely to odd integers $3, 5, 7, \ldots$ cf. [9]. Owing to (24) we can state

THEOREM 6. *Let $\nu \geq 0$ and $\rho > 2$ be fixed, $\nu \in boldZ$, $\rho \in boldR$. The function*

$$\aleph(m, 2; \nu, \rho) = (\rho + \nu + 1)^{-1} \log_2(1 + D^*(m, 2; \nu, \rho)^{-1}) \tag{30}$$

defined for all real $m > 1$, except for points from $S_{\nu\rho}$, represents an interpolation of the function period $k(2, m)$.

Indeed, for odd integers $m > 1$ we have $\aleph(m, 2; \nu, \rho) = k(2, m)$.

Case 2.2. In order to reduce the lower bound of ν by 1, and also to create a possibility of imposing an infinite number of conditions on the coefficients a_n given by (4), we suggest the use of the following distribution

$$\varphi(x) = \sum_{\mu=0}^{\infty} A_\mu \psi_\mu(x) \tag{31}$$

$$\psi_\mu(x) = \pi m^{-1} \sum_{n=-\infty}^{\infty} (-1)^n \delta(x - \pi m^{-1}(n + \beta_\mu)) \tag{32}$$

where δ denotes 6-distribution, $(\beta_\mu)_\mu$ is an arbitrary sequence of real numbers from the segment $[a, b]$ contained in the interval $[0, 2)$, $(A_\mu)_\mu$ is a sequence of indeterminates which could take complex values. The distribution (32) is periodic, with the period $2\pi/m$ and can be expressed by Euler polynomials

$$\psi_\mu(x) = 2^{-1}(\overline{E}_0(2^{-1}(m\pi^{-1}x - \beta)) - \overline{E}_0(2^{-1}(m\pi^{-1}x - \beta_\mu - 1))) \tag{33}$$

where $\overline{E}_0(x) = \overline{E}_1{}'(x)$, derivative being distributional. If coefficients A_μ in (31) are chosen so that the series converges distributionaly then it represents a periodic distribution with the period $2\pi/m$. A sufficient condition for that is

$$A_\mu = O(\mu^{-1-\varepsilon}) \tag{34}$$

where $\varepsilon > 0$ is arbitrary. In the sequel we assume (34) satisfied. Series (31) remains convergent when every term is multiplied by the function $\exp(jmxi)$ of the class C^∞. Such series of periodic distributions can be integrated term by term within one period (cf. [p. 59, 4.2.2., 1]) so that we obtain corresponding Fourier expansion

$$\varphi(x) = \sum_{j\in boldZ}{}' \alpha_j \exp(-jmxi), \qquad \alpha_j = \sum_{\mu=0}^{\infty} A_\mu \exp(j\pi\beta_\mu i). \tag{35}$$

Owing to (34) we find that $|\alpha_j| < \sum_\mu |A_\mu|$. Multiplying the left-hand side of (10) by $\varphi(x)\exp(-ix)$ we obtain distributionally convergent series

$$L(x) = \sum_{j\in boldZ}{}' \alpha_j \exp(-(jm - h(s))xi) \tag{36}$$

which represents periodic distribution with the period 2π. We can integrate this series within one period, to obtain right-hand equality in (6). Next we shall need a simple lemma from the theory of distributions.

Lemma 7. *Let there be given series of functions of class C^∞: $\omega(x) = \sum_{n=0}^\infty \omega_n(x)$ convergent in the interval (a,b), and let distribution $f(x)$ is of order k in the same interval. (Notion order in the sense of [p. 65, 1].) If the series $\sum_{n=0}^\infty \omega_n^{(k)}(x)$ uniformly converges in (a,b), then $\omega(x)f(x) = \sum_{n=0}^\infty \omega_n(x)f(x)$ and the series is distributionally convergent.*

Proof of Lemma 7 is simple, and therefore it is omitted here.

Now we take the restriction of $\varphi(x)$ to the interval $(0, 2\pi)$:

$$\varphi_1(x) = \pi m^{-1} \sum_{\mu=0}^{\infty} A_m u \sum_L (-1)^r \delta(x - z_\mu(r))$$

where $z_\mu(r) = \pi m^{-1}(r + \beta_\mu)$ and the range of summation L is defined by $0 < z_\mu(r) < 2\pi$, $r \in \mathbf{Z}$, and multiply the right-hand side of (10) by $\varphi_1(x)\exp(-ix)$. Since all conditions for application of Lemma 7 are fulfilled, we arrive at the distribution

$$(-1)^a 2^{1/2} \sum_{n=0}^{\infty} b_n \chi_n(x) \tag{37}$$

where b_n is given by (11) and

$$\chi_n(x) = \pi m^{-1} \sum_{\mu=0}^{\infty} A_\mu \sum_L (-1)^r P_n(\pi^{-1} z_\mu(r) - 1) \exp(-i z_\mu(r)) \delta(x - z_\mu(r)) \tag{38}$$

is distribution on $(0, 2\pi)$, $n = 0, 1, 2, \ldots$. In the next step we construct the periodic extension of the distribution (38) over the whole $\mathbf{R}$:

$$\chi_n^*(x) = \pi m^{-1} \sum_{\mu=0}^{\infty} A_\mu \sum_L (-1)^r P_n(\pi^{-1} z_m u(r) - 1) \exp(-i z_\mu(r)) \cdot \\ \cdot \sum_{\lambda=-\infty}^{\infty} \delta(x - z_\mu(r) + 2\lambda\pi) \tag{39}$$

and we consider the series

$$R(x) = (-1)^a 2^{1/2} \sum_{\mu=0}^{\infty} b_n \chi_n^*(x), \tag{40}$$

which is distributionally convergent on $\mathbf{R}$.

Indeed, there are functions $F_{\mu,n}(x)$ continuous on $\mathbf{R}$ such that the series

$$G_n(x) = \sum_{\mu=0}^{\infty} A_\mu F_{\mu,n}(x)$$

converges locally uniformly on $\mathbf{R}$ and $G_n''(x) = \chi_n^*(x)$. Also the functions $G_n(x)$ are continuous on $\mathbf{R}$. By the properties of Legendre polynomials and some other reasons one can show that for every finite segment $a \le x \le b$ the esimate $F_{\mu,n}(x) = O(1)$ holds uniformly with regard to μ and n. It follows that $G_n(x) = O(1)$ for $a \le x \le b$ and uniformly with regard to n. Therefore the series

$$\sum_{n=0}^{\infty} b_n G_n(x)$$

is locally uniformly convergent on $\mathbf{R}$ (by the properties of Bessel functions), and our statement concerning series (40) is proved.

Since periodic distributions $L(x)$, cf. (36), and $R(x)$, cf. (40), are equal on $(0, 2\pi)$, and there are neighborhoods of 0 and 2π in which both distributions coincide with 0, we conclude that $L(x) = R(x)$ on $\mathbf{R}$. Integration of this equality within the period $(0, 2\pi)$ yields

$$(-1)^a 2^{1/2} \sum_{n=0}^{\infty} a_n b_n = \int_0^{2\pi} L(x) dx \tag{41 }$$

where

$$a_n = \int_0^{2\pi} \chi_n(x) dx = \sum_{\mu=0}^{\infty} A_\mu S_{\mu,n}, \tag{42}$$

$$s_{\mu,n} = \pi m^{-1} \sum_L (-1)^r P_n(\pi^{-1} z_\mu(r) - 1) \exp(-i z_\mu(r)), \tag{43}$$

b_n is defined by (11) and the right-hand side of (41) is given by the right-hand equality in (6). From (42) and (43) we see that a_n is bounded with regard to n, so that further considerations could follow the ideas employed in the Case 2.1. However, before continuing in this direction, we may apply Poisson-Mordell theorem [p. 74, 7] or [5] and Fourier transform of Legendre polynomials to modify the expression (43). After some calculations the result reads

$$s_{\mu,n} = \pi i^{-n} 2^{1/2} \sum_{j\in\mathbf{Z}}{}' \exp(j\pi\beta_m ui)(mj+1)^{-1/2}_{Jn+1/2}(\pi(mj+1)). \tag{44}$$

Similarly as in the Case 2.1. (41) is multiplied by the weighting function $a^{-\rho s}$ and summation over $s \in \mathbf{N}$ yields

$$\sum_{n=0}^{\infty} i^n (n+1/2) a_n c_n = 2\pi \sum_{\mu=0}^{\infty} A_\mu \lambda_\mu \tag{45}$$

where

$$\lambda_\mu = \sum_{u=1}^{\infty} a^{-\rho uk} \exp(h(uk)\pi\beta_\mu i/m), \tag{46}$$

and c_n is given by (25). Substituting by sequence $(\beta_\mu)_\mu$ by $(2-\beta_\mu)_\mu$ it is possible to substitute $\exp(j\pi\beta_\mu i)$ in (44) and $\exp(h(uk)\pi\beta_\mu i/m)$ in (46) by $w(j\pi\beta_\mu i)$ and $w(h(uk)\pi\beta_\mu i/m)$ respectively, where w stands for sine or cosine function. We are interested here in the particular case $A_0 = l$, $A_\mu = 0$ for $\mu = 0$, $\beta_0 = 1$, $a = 2$, m odd integer. Then (42) and (44) reduce to

$$a_n = -\pi i^{-n} 2^{1/2} \sum_{j\in\mathbf{Z}}{}' (mj+1)^{-1/2} J_{n+1/2}(\pi mj+1)), \tag{47}$$

while the right-hand side of (45) is expressible in the finite form $-2\pi(a^{\rho k}-1)^{-1}$. This particular distributional proof showed that the relation (24) together with (25) and (16) holds also for $\nu = -1$.

REFERENCES

[1] P. Antosik, Ja. Mikusińskií, R. Sikorskií, *Teorija obobščennyh funkcĭ*, Mir, Moskva, 1976 (translation from English).

[2] A. Erdélyi et al., *Higher Transcendental Functions, Vol II.* Mc Graw-Hill, New York, 1953.

[3] [3] A. Erdélyi et al., *Tables of Integral Transforms, Vol. I*, Mc Graw-Hill, New York, 1954.

[4] E.W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series, Vol. II*, Dover, 1957.

[5] L.J. Mordell, *Some applications of Fourier series in the analytic theory of numbers*, Proc. Cambridge Phil. Soc. **24** (1928), 585-596.

[6] N.E. Nörlund, *Vorlesungen über Differenzenrechnung*, Chelsea, New York, 1954.

[7] H. Rademacher, *Topics in Analytic Number Theory*, Springer, Berlin, 1973.

[8] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge, 1966.

[9] A. Žepić, *Numerička i stohastička obrada aritmeticke funkcije eksponent*, doktorska disertacija, Zagreb 1982.

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