PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 41 (55), 1987, pp. 97–110

# MIXED NORM SPACES OF ANALYTIC AND HARMONIC FUNCTIONS, II

**M. Pavlović**

**Abstract.** In this paper we continue the study of the spaces $h(p,q,\varphi)$ and $H(p,q,\varphi)$. We apply the main results of Part I to obtain new information on the coefficient multipliers of these spaces. For example, we find the multipliers from $h(p,q,\varphi)$ to $h(\infty,q_0,\varphi)$ for any $p \geq 1$, $q, p_0 > 0$ and any quasi-normal function $\varphi$, and this improves and generalizes a result of Shields and Williams [16]. We also describe the multipliers from $H(p,q,\alpha)$, $p \leq 1$, to $H(p_0,q_0,\alpha)$, $p_0 \geq p$, and $l^s$, $s > 0$.

## 0. Introduction

Let $h(U_R)$ be the class of all complex-valued harmonic functions in the disc $U_R = \{z : |s| < R\}$, $R > 0$. For a set $E$ of integers let $h_E(UR) = \{f \in h(UR) : \operatorname{supp}(\hat{f}) \subset E\}$. An $A$-space $X$ is a quasi-normed space satisfying the following conditions: 1. There exists a set $E$ such that $h_E(UR) \subset X \subset h_E(U)$ for all $R > 1 (U = U_1)$; 2. If $f \in X$ and $\zeta \in \overline{U}$, then $\|f_\zeta\| \leq \|f\|$ where $f_\zeta$ is defined by $f_\zeta(z) = f(\zeta z)$; 3. Let $P_r(f) = \|f_r\|$, $f \in h_E(U)$, $0 < r < 1$. Then the family $\{P_r\}$ defines a topology on $h_E(U)$, which coincides with the topology of uniform convergence on compact subsets of $U$.

If $X$ is complete, then the third condition may be replaced by the requirement that $X$ is continuously embedded into $h_E(U)$. (This can be proved by using the closed graph theorem.) In Part I, $A$-spaces are defined in a different way, but it is easily shown that the two definitions are equivalent.

A function $\varphi : (0,1) \to (0,\infty)$ is said to be quasi-normal if it is increasing, absolutely continuous, $\varphi(0+) = 0$ and $\varphi(2t)/\varphi(t) \leq C < \infty$ for $0 < t < 1/2$. If, in addition, $\varphi(at) \leq \varphi(t)/2$, $t > 0$, for some $a > 0$ then $\varphi$ is said to be normal. In Part I we defined the scale of spaces $X(q,\varphi)$ in the following way:

$X(q,\varphi)$ consists of all $f \in h_E(U)$ for which the function $F(r) := \varphi(1-r)\|f_r\|_X$, $0 < r < 1$, belongs to the Lebesgue space $L^q(m_\varphi)$, where $\mathrm{d}m_\varphi(r) = \varphi'(1-r)\mathrm{d}r/\varphi'(1-r)$.

---

*AMS Subject Classification* (1980): Primary 46E15, 30H05.

It was shown that $X(q,\varphi)$ is a complete $A$-space with the quasinorm

$$\|f\|_{X(q,\varphi)} = \|F\|_{L^q(m\varphi)}.$$

Throughout the paper we shall suppose that $\varphi$ is extended to the interval $(0,\infty)$ so that the following holds: 1. $\varphi(t)\varphi(1/t) \sim 1$, $t > 0$, i.e. $0 < c \le \varphi(t)\varphi(1/t) \le C < \infty$; 2. $1/\varphi$ is convex on $(1,\infty)$ and $\varphi(1+) = \varphi(1)$. Such an extension is possible; for example

$$\varphi(1)/\varphi(t) = \int_0^1 r^{t-1}\varphi'(1-r)dr, \ \ t > 1.$$

In this part we consider coefficient multipliers from $X(g,\varphi)$ to $Y(g_0,\psi)$. Multiplier problems for various spaces of analytic and harmonic functions have been considered by many authors. See, for example, [1, 2, 3, 4, 12, 15, 16]. Mainly, these results concern the spaces $H(p,q,\varphi){:=}\,H^p(q,\varphi)$ and $h(p,g,\varphi){:=}\,h^p(q,\varphi)$ with $\varphi(t) = t^\alpha$, where $H^p$ and $h^p$ are Hardy and harmonic Hardy spaces, respectively. (In this case we write a instead of $\varphi$.) The following result of Hardy and Littlewood [8] and Flett [5, 6] is one of the most important results in this area.

Theorem HLF. *If $0 < p,q \le \infty$ and $0 < \alpha,\beta < \infty$ then the operator $D^{\beta-\alpha}$ acts as an isomorphism from $H(p,q,\alpha)$ onto $H(p,q,\beta)$.*

The operator $D^s : h(U) \to h(U)(-\infty < s < \infty)$ is defined by

$$(D^s f)\hat{}\,(k) = (|k|+1)^s \hat{F}(k), \qquad -\infty < k < \infty.$$

In Section 1 we give some extensions of Theorem HLF. For example, if $\varphi$ is a normal function and $\alpha > 0$, then $H(p,q,\varphi)$ and $H(p,q,\alpha)$ are isomorphic via a multiplier transform. However, this transform is more complicated than in the case of Theorem HLF, and is not indepedent of $p$.

The multipliers from $h(\infty,\infty,\varphi)$ into itself, where $\varphi$ satisfies additional restrictions on regularity of growth, were described by Shields and Williams [16]. In Section 3 we describe the multipliers from $h(p,q,\varphi)$ to $h(\infty,q_0,\varphi)$, $p \le 1$, for any quasi-normal function $\varphi$. Using this we solve Problem $B$ of [16].

It was shown by Duren and Shields [4] that $g$ is a multiplier from $H(1,1,\alpha)$ into itself if and only if $M_1(r,g') \le C/(1-r)$, $0 < r < 1$ . We generalize this by finding the multipliers from $H(p,q,\alpha)$ to $H(p_0,q_0,\alpha)$, where $p \le \min(1,p_0)$.

In Section 5 we briefly discuss the multipliers from $H(p,q,\alpha)$ to the sequence space $l^s$. Some partial solutions to this problem are given by Ahern and Jevtić [1], Mateljević and Pavlović [11, 12] and others. (See [1, 12] for information and references). Here we consider the case $p \le 1$ and find the multipliersfor any $q > 0$ and $s > 0$. In the case $p \ge 2$ a stronger is known [2, 11].

Our method is based on the main result of Part I, which enables us to reduce the multipliers from $X(q,\varphi)$ to $Y(q_0,\varphi)$ to those from $X$ to $Y$. For our purposes it is convenient to introduce the spaces $X[q,W]$ in the following way.

Let $N$ be a non-negative integer, and let $\bigwedge := \{\lambda_n\}_0^\infty$ be an increasing sequence of positive integers. For a sequence $W := \{w_n\}_0^\infty$ or harmonic polynomials we write $W \in (N, \bigwedge)$ if the following conditions are satisfied:

$$f(z) = \sum_{n=0}^{\infty} w_n * f(z), \quad f \in h(U), \quad z \in U,$$

where the series is uniformly convergent on compact subsets of $U$;

$$\hat{w_n}(k) = 0 \text{ if } |k| \notin [\lambda_{n-1}, \lambda_{n+N}) \geq 0,$$

where $\lambda_{-1} = 0$.

We define $X[q, W] = \{f \in h_E(U) : \|f\|_{X[q,W]} < \infty\}$, where

$$\|f\|_{X[q,W]} = \sum_0^\infty \|w_n * f\|_X^q, \quad q < \infty,$$
$$\|f\|_{X[\infty,W]} = \sup_n \|w_n * f\|_X.$$

These spaces are generalizations of the sequence spaces $l(p, q)$ introduced by Kellogg [10]. One can prove that $X[q, W]$ are $A$-spaces. Their main properties will be given in Sections 1 and 2.

## 1. Isomorphisms between $X(q, \varphi)$ and $X[q, W]$

By Proposition 3.2, Part I, there exists a lacunary sequence $\bigwedge = \{\lambda_n\}_0^\infty$ of positive integers such that $\varphi(\bigwedge) := \{\varphi(\lambda_n)\}_0^\infty$ is normal, i.e.

$$c_1(1 + c)^j \varphi(\lambda_n) \leq \varphi(\lambda_{n+j}) \leq C^j \varphi(\lambda_n), \quad j, n \geq 0.$$

where $c_1, c, C$ are positive constants.

For an $A$-space $X$ let $s(X) = h_E(U)$, where $E$ is the unique set of integers such that $h_E(UR) \subset X \subset h_E(U)$, $R > 1$. Let $\mathcal{B}_N$ be the class consisting of all normed $A$-spaces and all $H^p$ with $p \geq l/N$.

THEOREM 1. *Let $N \geq 1$, let $\bigwedge$ be a lacunary sequence, and let the sequence $\varphi(\bigwedge)$ be normal. Then there exists a sequence $W \in (N, \bigwedge)$ and a function $g \in h(U)$ such that for all $X \in \mathcal{B}_N$ the following assertions hold:*

a) *The operator $g^*$ defined by $g^*(f) = f * g$ is an isomorphism of $X(q, \varphi)$ onto $Xq, [W]$.*

b) *$\|w_n * f\|_X \leq K \|f\|_X$ for $n \leq 0, f \in X$, where $K$ is independent of $X \in \mathcal{B}_N$.*

c) *$\hat{g}(n) = \hat{g}(-n) \sim 1/\varphi(n + 1)$, $n \geq 0$. If $N = 1$, then one can take $\hat{g}(n) = 1/\varphi(|n| + 1)$.*

Define the operators $D^\psi$ and $D_\varphi : h(U) \to h(U)$ by

$$(D_\varphi f)\hat{}\,(n) = \hat{f}(n)/\varphi(|n| + 1), \;\; (D^\psi f)\hat{}\,(n) = \psi(|n| + 1)\hat{f}(n).$$

The following theorem generalizes the case $\geq 1$ of Theorem HLF. It is an immediate consequence of Theorem 1.1.

THEOREM 1.2. *Let $n$ be a lacunary sequence such that both $\varphi(\bigwedge)$ and $\psi(\bigwedge)$ are normal. Then $D_\varphi D^\varphi$ acts as an isomorphism from $h(p,q,\varphi)$, $p \geq 1$, onto $h(p,q,\psi)$.*

*Remark.* The function $\psi$ is extended to $(0,\infty)$ in the same way as $\varphi$.

THEOREM 1.3. *(with the hypotheses of Theorem 1.2.). For every $p > 0$ there exists an analytic function $g$ such that $\hat{g}(n) \sim \psi(n+1)/\varphi(n+1)$, $n \geq 0$, and the operator $g^*$ is an isomorphism from $H(p,q,\varphi)$ onto $H(p,q,\psi)$.*

If $\varphi$ is normal, then the sequence $\{\varphi(2^n)\}_0^\infty$ is normal. Thus we have

COROLLARY 1.1. *If $\varphi$ is a normal function and $\alpha$ is a positive number, then $H(p,q,\varphi)$ and $H(p,q,\alpha)$ $(p > 0)$ are isomorphic via a multiplier transform.*

*Proof of Theorem* 1.1. By Theorems 2.1. and 4.1. (Part I) and their proofs, there is $W \in (N, \bigwedge)$ such that $(b)$ holds, $\hat{w}_n(-k) = \hat{w}_n(k)$ and

$$\|f\|_{X(q,\varphi)} \sim \left\{ \sum_0^\infty [\varphi(1/\lambda_n)\|w_n * f\|_X] \right\}^{1/q}, \quad f \in s(X).$$

Thus it suffices to find a function $g$ independent of $X \in \mathcal{B}_N$ and satisfying $(c)$ and

$$\|w_n * g * f\| \sim \varphi(1/\lambda_n)\|w_n * f\|, \quad f \in s(X), \quad n \geq 0 \tag{1.1}$$

Let $B_n = \varphi(1/\lambda_n)^{1/m}$, $n \geq 0$, where $m$ is a positive integer which will be chosen later on. Define the functions $g_1, \dots, g_m$ in the following way:

$$g_1(z) = \sum_0^\infty B_n w_n(z), \quad g_j = g_1 * g_{j-1}, \quad 2 \leq 1 \leq m.$$

We have

$$w_n * g_1 = B_n \sum_{k=0}^\infty w_n * w_k + \sum_{k=0}^\infty (B_k - B_n) w_n * w_k = B_n w_n + \sum_{k=n-N}^{n+N} (B_k - B_n) w_n * w_k,$$

where $B_k = B_o$ and $w_k = 0$ for $k < 0$, Using the triangle inequality for $\|\cdot\|_X^s$, where $s = 1/N$, we obtain

$$\|w_n * g_1 * f\|^s \geq B_n^s \|w_n * f\|^s - \sum k = n - N^{n+N} |B_k - B_n|^s \|w_n * w_k * f\|^s, \quad f \in s(X).$$

Hence, by $(b)$,

$$\|w_n * g_1 * f\|^s \geq B_n^s \|w_n * f\|^s - K^s \|w_n * f\|^s \sum_{k=n-N}^{n+N} |B_k - B_n|^s.$$

Since $\varphi(\bigwedge)$ is normal, there exists $b \in (0,1)$ such that $\varphi(1/\lambda_n + N) \geq b\varphi(1/\lambda_n)$ for all $n \geq 0$. Using this we get

$$\begin{aligned}\sum_{k=n-N} 6n + N|B_k - B_n|^s &\leq (B_n - N_{n+N})^s N + (B_{n=N} - B_n)^s N \\ &\leq NB_n^s(1 - b^{1/m}) + NB_n^s(b^{-1/m} - 1)^s.\end{aligned}$$

Choose $m$ so that

$$NK^s(1 - b^1/m)^s + (b^{-1/m} - 1)^s \leq 2^{-s}. \tag{1.2}$$

Then $\|w_n * g_1 * f\| \geq 2^{-1}B_n\|w_n * f\|$ and, by induction,

$$\|w_n * g_m * f\| \geq 2^{-m}\varphi(1/\lambda_n)\|w_n * f\|. \tag{1.3}$$

In the other direction, from the identity

$$w_n * g_1 * f = \sum_{k=n-N}^{n+N} B_k w_n * w_k * f$$

we obtain

$$\|w_n * g_1 * f\|^s \leq K^s\|w_n * f\|^s \sum_{k=n-N}^{n+N} B_k^s \leq K_s\|w_n * f\|^s(2N+1)B_{n-N}^s.$$

This implies

$$\|w_n * g_m * f\| \leq C\|w_n * f\|, \quad f \in s(X),\ n \geq 0, \tag{1.4}$$

where $C$ is a positive real constant.

In order to estimate the coefficients of $g_m$ observe that $\hat{g}_m(k) = \hat{g}_1(k)^m$. Thus we have to prove $\hat{g}_1(k) \sim \varphi(k+1)^{-1/m}$, $k \geq 0$. It is easily verified that $\hat{g}_1(k) = B_0$ for $k < \lambda_0$. If $k \geq \lambda_0$ then we choose $n \geq 0$ so that $\lambda_n \leq k < \lambda_{n+1}$. Then

$$\hat{g}_1(k) = B_{n+1} + \sum_{j=n-N}^{n} (B_j - B_{n+1})\hat{w}_j(k).$$

Taking $X = H^\infty$ and $f(z) = z^k$ in $(b)$ we see that $|w_j(k)| \leq K$ for all $j, k \geq 0$, where $K$ is the same as in (1.2). Hence

$$\begin{aligned}|\hat{g}(k)| &\geq B_{n+1} + \sum_{j=n-N}^{n} (B_j - B_{n+1}) \\ &\leq B_{n+1} - K(B_{n-N} - B_{n+1})N \\ &\leq B_{n+1} - KB_{n+1}(B_{n-N}/B_{n+N} - 1)N \\ &\leq B_{n+l} - KB_{n+1}(b^{-2/m} - 1)N \geq 2^{-1}B_{n+1},\end{aligned}$$

where $m$ is chosen so that the inequalities (1.2) and $K(b^{-2/m}-1)N \le 1/2$ hold. This proves that $\hat{g}_m(k) \ge c\varphi(k+1)^{-1}$. The proof of the inequality $\hat{g}_m(k) \le C\varphi(k+1)^{-1}$ is simple. Taking $g = g_m$ we see that the condition (1.1) is satisfied.

In the case of normed spaces the result follows from Lemma 5.3 of Part I.

## 2. Multipliers from $X[q,W]$ to $Y[q,W]$

Let $X, Y$ be $A$-spaces, $s(X) \cap s(Y) \ne \emptyset$. A function $g \in h(U)$ is a multiplier from $X$ to $Y$ if the map $f \mapsto f * g$ is a bounded linear operator from $X$ to $Y$. If the spaces are complete then this is equivalent with the requirement that $f * g \in Y$ for all $f \in X$. In Part I we have defined the space

$$(X \to Y) = \{g \subset s(X) \cap s(Y) : g \text{ is a multiplier from } X \text{ to } Y\},$$

with the quasi-norm

$$\|g\|_{X,Y} = \sup\{\|f * g\|_Y : f \in X,\ \|f\|_X \le 1\}.$$

We shall prove that there is a simple connection between the spaces $(X[q,W] \to Y[q_0,W])$ and $(X \to Y)$ provided that $\|w_n * f\|_X \le C\|f\|_X$, i.e., $X \subset X[\infty, W]$. One can prove that all these spaces are $A$-spaces.

Throughout this section we suppose $O < q$, $q_0 \le \infty$ and

$$q_1 = \begin{cases} \infty & \text{if } q \leqq q_0, \\ qq_0/(q-q_0) & \text{if } q > q_0 \end{cases}$$

THEOREM 2.1. *Let $X \subset X[\infty, W]$ where the inclusion is continuous. Then*

$$(X[q,W] \to Y[q_0,W]) = (X \to Y)[q_1, W]. \tag{2.1}$$

*Proof.* Since $\hat{w}_n(k) = 0$ for $|k| \notin [\lambda_{n-1}, \lambda_{n+N})$ we have

$$w_n * w_j = 0 \text{ for } |j-n| \ge N+1. \tag{2.2}$$

Let

$$P_n = \sum_{j=n-N}^{n+N} w_j, \quad n \ge 0,$$

where $w_j = 0$ for $j < 0$. From (2.2) and the identity $f = \sum w_n * f$ it follows that

$$P_n * w_n = w_n, \quad n \ge 0, \tag{2.3}$$

$$w_n * P_j = 0 \text{ if } |j-n| \ge 2N+1. \tag{2.4}$$

Let $g \in Z[q_1] = Z[q_1, W]$, $Z = (X \to Y)$. In view of (2.3) and the definition of $(X \to Y)$ we have

$$\|w_n * f * g\|_Y = \|P_n * f * w_n * g\|_Y \le \|P_n * f\|_X \|w_n * g\|_Z.$$

By Hölder's inequality

$$\|f * g\|_{Y[q_0]} \leq \|\{A_n\}\|_{l^q} \|g\|_{Z[q_1]},$$

where $A_n = \|P_n * f\|_X$, $n \geq 0$. Using the inequality

$$\|P_n * f\| \leq C \sum_{j=n-N}^{n+N} \|w_j * f\|$$

and Lemma 5.2 of Part I we find $\|\{A_n\}\|_{l^q} \leq C\|f\|_{X[q]}$. This concludes the proof of the inclusion $Z[q_1] \subset (X[q] \to Y[q_0])$.

Let $g \in (X[q] \to Y[q_0])$. Then

$$\|f * g\|_{Y[q_0]} \leq C\|f\|_{X[q]}, \quad f \in X[q].$$

Let $l^q(X)$ be the space of those sequences $F = \{f_n\}_0^\infty$ for which $f_n \in X$, $n \geq 0$ and

$$\|F\|_{l^q(X)} := \|\{\|f_n\|_X\}\|_{l^q} \leq \infty.$$

Let $\overline{l^q}(X)$ be the subspace of $l^q(X)$ consisting of $\{f_n\}$ such that $f_n = 0$ for $n$ large enough. Define the operators $V_m$, $0 \leq m \leq 2N$, on $\overline{l^q}(X)$ by

$$V_m F = \sum_{n=0}^{\infty} P_{n,m} * f_n,$$

where $P_{n,m} = P_{(2N+1)n+m}$. Using the hypothesis $X \subset X[\infty]$ and the relation (2.4) one shows that $V_m's$ are bounded linear operators from $\overline{l^q}(X)$ to $X[q]$. (See Lemma 5.1, Part I.) It follows that

$$\|(V_m F) * g\|_{Y[q_0]} \leq C\|F\|_{l^q(x)}, \quad F \in \overline{l^q}(X), \quad 0 \leq m \leq 2N,$$

where $C$ is independent of $F, m$. This implies

$$\left\{ \sum_{k=0}^{\infty} \|w_{k,m} * (V_m F) * g\|_Y^{q_0} \right\}^{1/q_0} \leq C\|F\|_{l^q(X)},$$

where $w_{k,m} = w_{(2N+1)k+m}$. If $k \neq n$ then $|(2N+1)k + m - (2N+1)n - m| = (2N+1)k - n \geq 2N+1$, and this implies $w_{k,m} * P_{k,m} = 0$, by (2.4). Hence

$$w_{k,m} * (V_m F) = w_{k,m} * P_{k,m} * f_k = w_{k,m} * f_k.$$

(In the last step we have used (2.3).) Now we have

$$\left\{ \sum_{k=0}^{\infty} \|w_{k,m} * g * f_k\|_Y^{q_0} \right\}^{1/q_0} \leq C\|F\|_{l^q(X)}. \tag{2.5}$$

Fix $m$, $0 \leq m \leq 2N$, and $\varepsilon < 1$, and for every $k \geq 0$ choose $h_k \in X$ so that $\|h_k\|_X = 1$ and

$$\|w_{k,m} * g * h_k\|_Y \geq \varepsilon\|w_{k,m} * g\|_Z. \tag{2.6}$$

Putting $f_k = a_k h_k$, where $\{a_k\}_0^\infty \in \overline{l^q} = (R)$ ($R$ is the scalar field) we get from (2.5) and (2.6)

$$\varepsilon\left\{\sum_{k=0}^{\infty}[|a_k|\|w_{k,m} * g\|_Z]^{q_0}\right\}^{1/q_0} \le C\left\{\sum_{k=0}^{\infty}|a_k|^q\right\}^{1/q},$$

where $C$ is independent of $\varepsilon$, $m$ and $\{a_k\}$. This gives

$$\left\{\sum_{k=0}^{\infty}\|w_{k,m} * g\|_Z^{q_1}\right\}^{1/q_1} < \infty$$

for all $m$, $0 \le m \le 2N$. If $q_1 < \infty$ then

$$\sum_{n=0}^{\infty}\|w_n * g\|^{q_1} = \sum_{m=0}^{2N}\sum_{k=0}^{\infty}\|w_{k,m} * g\|^{q_1} < \infty$$

whence $g \in Z[q_1]$; similarly for $q_1 = \infty$. This completes the proof of Theorem 2.1.

As a consequence of Theorems 1.1 and 2.1 we have

THEOREM 2.2. *If $Z = (X \to Y)$, where $X$, $Y$ are normecl spaces, then*

$$(X(q,\varphi) \to Y(q_0,\varphi)) = \{g \in s(X) \cap s(Y : D^\varphi g \in Z(q_1,\varphi)\}.$$

*Proof.* It follows from Theorem 1.1 that $(X(q,\varphi) \to Y(q_0,\varphi)) = (X[q,W] \to Y[q_0,W])$ for a suitable $W \in (1,\wedge)$. Now the desired result is obtained by using Theorem 2.1 and then Theorem 1.1 and the fact that $Z$ is a normed space.

## 3. Multipliers of $h(p,g,\varphi)$

In this section we apply the preceding results to the case of the spaces $h(p,q,\varphi) = h^p(q,\varphi)$. Note that if $p \ge 1$ then $\|f_r\|_{h_p} = M_p(r,f)$, $O < r < 1$, so that $f \in h(p,q,\varphi)$ if and only if

$$\int_0^1 [\varphi(1-r)M_p(r,f)]^q dm_\varphi(r) < \infty.$$

THEOREM 3.1. *Let $q$, $q_0$, $q_1$, be as in Section 2, let $p \ge 1$ and $1/p + 1/p' = 1$. For a function $g$ the following are equivalent.*

(i) *$g$ is a multiplier from $h(1,q,\varphi)$ to $h(p,q_0,\varphi)$;*

(ii) *$g$ is a multiplier from $h(p',q,\varphi)$ to $h(\infty,q_0,\varphi)$;*

(iii) $D^\varphi g \in h(p,q_1,\varphi)$.

*Proof.* By Theorem 2.2. $(h(1,q,\varphi) \to h(p,q_0,\varphi))$ is the set of all $g \in h(U)$ such that $D^\varphi g \in (h^1 \to h^p)(q_1,\varphi)$. Since $(h^1 \to h^p) = (h^{p'} \to h^\infty) = h^p$ we see that (i) $\Leftrightarrow$ (iii). The proof. of (ii) $\Leftrightarrow$ (iii) is the same.

COROLLARY 3.1. *If $p \geq 1$ then the set*

$$M(p, \varphi) := (h(p, q, \varphi) \to h(p, q, \varphi))$$

*is independent of $q$. Furthermore*

$$M(\infty, \varphi) = M(1, \varphi) = \{g \in h(U) : D^{\varphi} g \in h(1, \infty, \varphi)\}. \tag{3.1}$$

Shields and Williams [16] proved that (3.1) holds provided that *varphi* satisfies some regularity conditions.

The set $M(p, \varphi)$ is an algebra with unit. It follows from Theorem 2.2 and the equality $(h^p \to h^p) = (h^{p'} \to h^{p'})$ that $M(p, \varphi) = M(p', \varphi)$. It is clear that $M(2, \varphi) = \{g \in h(U) : \hat{g}$ is bounded$\}$. Concerning the set $M(p, \varphi)$ we can only prove that it increases with $p \in [1, 2]$.

PROPOSITION 3.1. *If $1 \leq p \leq s \leq 2$ then $M(p, \varphi) \subset M(s, \varphi)$.*

*Proof.* It is trivial to check that $(h^p \to h^p) \subset (h^2 \to h^2)$. By the Riesz-Thorin theorem, $(h^p \to h^p) \subset (h^s \to h^s)$ if $p \leq s \leq 2$. Now if $g$ is in $M(p, \varphi)$ then by Theorem 2.2, $D^{\varphi} g \in (h^p \to h^p)(\infty, \varphi) \subset (h^s \to h^s)(\infty, \varphi)$, and this concludes the proof.

In [16] Shields and Williams posed the question: If the spaces $h(\infty, \infty, \varphi)$ and $h(\infty, \infty, \psi)$ have the same set of multipliers are they isomorphic via a multiplier transform? The answer is yes, as the following theorem shows.

THEOREM 3.2. *If $M(\infty, \varphi) = M(\infty, \psi)$ then the operator $D_{\varphi} D^{\psi}$ acts as an isomorphism from $h(\infty, q, \varphi)$ onto $h(\infty, q, \psi)$.*

*Proof.* By Theorem 1.2, it is enough to find a lacunary sequence $\{\lambda_n\}$ such that both $\{\varphi(\lambda_n)\}$ and $\{\psi(\lambda_n)\}$ are normal. Let

$$g_1(z) = \sum_0^{\infty} z^{t_n} \text{ and } g_2(z) = \sum_0^{\infty} z^{s_n}, \quad z \in U,$$

where $\{t_n\}$ and $\{s_n\}$ are lacunary sequences of integers such that $\{\varphi(t_n)\}$ and $\{\psi(s_n)\}$ are normal. By the well-known fact on lacunary trigonometric series,

$$M_1(r, D^{\psi} g_2) \sim \left\{ \sum_0^{\infty} \psi(s_n)^2 r^{2 s_n} \right\}^{1/2}, \quad 0 < r < 1.$$

Hence, by Lemma 3.1 of Part I, $M_1(r, D^{\psi} g_2) \leq C/\psi(1-r)$, i.e., $D^{\psi} g_2 \in h(1, \infty, \psi)$. Therefore $g_2 \in M(\infty, \psi)$, by (3.1 ). Using this and the hypothesis $M(\infty, \varphi) = M(\infty, \varphi)$ we conclude that $g_2 \in M(\infty, \varphi)$. Hence, by (3.1 ), $D^{\psi} g_2 \in h(1, \infty, \varphi)$, i.e.,

$$\left\{ \sum_0^{\infty} \varphi(s_n)^2 r^{2 s_n} \right\}^{1/2} \leq C/\varphi(1-r).$$

This implies
$$\sum_{T_n} \varphi(s_k)^2 \leq C\varphi(t_n)^2, \quad n \geq 0,$$
where $T_n = \{k : t_n \leq s_k < t_{n+1}\}$. Since $\varphi(t_n) leq \varphi(s_k) \leq \varphi(t_{n+1}) \leq C\varphi(t_n)$ for $k \in T_n$, we conclude that card $(T_n) \leq C < \infty$, $n \geq 0$. Using this and the analogous fact for the sets $\{k : s_n \leq t_k < s_{n+1}\}$ we find a positive integer $m$ such that for all $n \geq 0$, $j \geq 1$

(i) $$\operatorname{card}\{k : t_n \leq s_k < t_{n+j}\} \leq mj,$$

(ii) $$\operatorname{card}\{k : s_n \leq t_k < s_{n+j}\} \leq mj.$$

Put $\lambda_k = t_{km}$, $k \geq 0$. It is easy to see that the sequence $\{\varphi(\lambda_k)\}$ is normal. Let $k_0$ be such that $t_{k_0 m} \geq s_0$. If $k > k_0$ choose $n$ so that $s_n \leq t_{km} < s_{n+1}$. Then, by (ii), $\lambda_{k+j} = t_{km+jm} \geq s_{n+j}$ and, consequently,
$$\psi(\lambda_{k+j})/\psi(\lambda_k) \geq \psi(s_{n+j})/\psi(s_{n+1}) \geq ca^j,$$
where $a > 1$ and $c > 0$ are constants. (Here we have used the hypothesis that $\{\psi(s_n)\}$ is normal.) On the other hand, it follows from (i) that $s_{n+m+1} \geq t_{km+m} = \lambda_{k+1}$. Hence
$$\psi(\lambda_{k+1})/\psi(\lambda_k) \leq \psi(s_{n+m+1})/\psi(s_n) \leq C.$$
Thus the sequence $\{\psi(\lambda_k)\}$ is normal, what was to be proved.

As a further application of the equality (3.1) we have the following characterization of self-conjugate spaces. The space $h(p,q,\varphi)$ is said to be self-conjugate if
$$f \in h(p,q,\varphi) \text{ implies } \sum_{n=0}^{\infty} \hat{f}(n)z^n \in h(p,q,\varphi).$$
If $1 < p < \infty$ then $h(p,q,\varphi)$ is self-conjugate because of the Riesz theorem. Hardy and Littlewood [7, 9] proved that $h(p,\infty,\alpha)$ is self-conjugate for any $p > 0$. For further information see [5, 6].

Theorem 3.3. *For every $q \in (0,\infty]$ the following statements are equivalent.*

(i) $h(1,q,\varphi)$ *is self-conjugate;*

(ii) $h(\infty,q,\varphi)$ *is self-conjugate;*

(iii) $\varphi$ *is a normal function.*

*Proof.* Observe that $h(p,q,\varphi)$ is self-conjugate if and only if the function $\sum_0^\infty z^n$ belongs to $M(p,\varphi)$. Since $M(1,\varphi) = M(\infty,\varphi)$ we see that (i) $\Leftrightarrow$ (ii). Assuming (iii) we have to prove that $D^\varphi h \in H(1,\infty,\varphi)$, where $h(z) = 1/(1-z) = -\sum_0^\infty z^n$. By Theorem 1.2, this is equivalent with $D^1 h \in H(1,\infty,1)$. Since $D^1 h(z) = (1-z)^{-2}$ we have $M_1(r, D^1 h) = (1-r^2)^{-1}$, and this gives the desired result.

To prove that (i) implies (iii) we use the inequality
$$\|f\|_1 \geq \frac{1}{\pi}\sum_0^\infty (n+1)^{-1}|\hat{f}(n)|, \quad f \in H^1,$$

[3, Theorem 6.1]. In particular,

$$M_1(r, D^{\varphi}h) \geq \frac{1}{\pi}\sum_0^{\infty}(n+1)^{-1}\varphi(n+1)r^n, 0 < r < 1.$$

Thus if (i) holds then $\sum_0^{\infty}(n+1)^{-1}\varphi(n+1)r^n \leq c/\varphi(1-r)$. This implies

$$\varphi(\lambda_k)\sum_{n=\lambda_k}^{\lambda_{k+1}}(n+1)^{-1} \leq C/\varphi(1/\lambda_{k+1}), \quad k \geq 0,$$

where $\{\lambda_k\}$ is a lacunary sequence of integers such that $\varphi(\lambda_k) \sim 2^k$, i.e. $\varphi(1/\lambda_k) \sim 2^{-k}$. (See Proposition 3.2 Part I.) It follows that $\lambda_{k+1}/\lambda_k \leq C$, $k \geq 0$. By using this one shows that $\varphi^-(2t) \leq C\varphi^-(t)$, $t > 0$, where $\varphi^-$ is the inverse function. Hence $\varphi^-(t/C) \leq \varphi(t)/2$, and this concludes the proof of the theorem.

## 4. Multipliers from $H(p,q,\alpha)$ to $H(p_0,q_0,\alpha)$

Let $\alpha$ be a positive real number. A function $f \in H(U)$ (= the class of analytic functions) belongs to $H(p,q,\alpha) = H^p(q,\alpha)$ if and only if

$$\left\{\int_0^1 (1-r)^{q\alpha-1}M_p^q(r,f)dr\right\}^{1/q} < \infty.$$

If $q = \infty$ this should be read as

$$\sup_{0<r<1}(1-r)^{\alpha}M_p(r,f) < \infty.$$

The main results of this section is the following.

THEOREM 4.1. *Let $p \leq 1$, $p_0 \geq p$ and $0 < q$, $q_0 \leq \infty$. A function $g \in H(U)$ is a multiplier from $H(p,q,\alpha)$ to $H(p_0,q_0,\alpha)$ if and only if $D^{1/p}g \in H(p_0,q_1,1)$.*

Here, as before, $q_1 = \infty\,(q \leq q_0)$; $q_1 = q_0\,(q = \infty)$; $q_1 = qq_0/(q-q_0)$ if $q_0 < q < \infty$.

Note that $D^{1/p}g \in H(p_0,q_1,1)$ if and only if

$$\left\{\int_0^1 (1-r)^{q_1-1}M_{p_0}^{q_1}(r,d^{1/P}g)dr\right\}^{1/q_1} < \infty.$$

COROLLARY 4.1. *Let $p \leq 1$. Then $g$ is a multiplier from $H(p,q,\alpha)$ to itself if and only if $M_p(r,D^{1/p}g) \leq C/(l-r)$, $0 < r < 1$.*

This generalizes a result of Duren and Shields [4] ($p = q = 1$).

Let $N$ be a positive integer and choose a sequence $W = \{w_n\}_0^{\infty}$ of harmonic polynomials such that $W \in (N, \{2^n\}_1^{\infty})$ (see Introduction) and for all $p > 1/(N+1)$ and $q > 0$

$$\|f\|_{H(p,q,\alpha)} \sim \left\{\sum_0^{\infty}[2^{-n\alpha}\|w_p * f\|_p]^q\right\}^{1/q} \qquad f \in H(U),$$

where $\|\cdot\|_p$ stands for the norm of $H^p$. Since

$$w_n * f(z) = \sum_{j=2^{n-1}}^{2^{n+N}} \hat{w}_n(j)\hat{f}(j)z^j$$

we have, by Lemma 3.1 [11],

$$r^{2n+N}\|w_n * f\|_p \le M_p(r, w_n * f) \le r^{2n-1}\|w_n * f\|_p.$$

$(n \ge 1,\ 0 < r < 1)$. After elementary calculations it follows that

$$2^{-n\alpha}\|w_n * f\|_p \sim \|w_n * f\|_{H(p,s,\alpha)}, \quad f \in H(U), \quad n \ge 0,$$

where $s$ is an arbitrary positive number or $\infty$. Thus we have the following.

Theorem 4.2. *Let $p > 1/(N+1)$, $0 < q \le \infty$ and $0 < s \le \infty$. Then*

$$H(p,q,\alpha) = H(p,s,\alpha)[q,W].$$

Observe that the polynomials $w_n$ are independent of $p, q, s$.

Combining Theorems 4.2 and 2.1 wc get the identity

$$(4.1) \qquad (H(p,q,\alpha) \to H(p_0,q_0,\beta)) = (H(p,\infty,\alpha) \to H(p_0,\infty,\beta))[q_1,W],$$

which shows that the general case of Theorem 4.1 follows from the special case $q_0 = q = \infty$.

*Proof of Theorem* 4.1. Let $g$ be a multiplier from $H(p,\infty,\alpha)$ to $H(p_0,\infty,\alpha)$. Then

$$\|g * f_r\|_{H(p_0,\infty,\alpha)} \le C\|f_r\|_H(p,\infty,\alpha), \quad 0 < r < 1,$$

where $f(z) = \sum_{n=0}^{\infty}(n+1)^{m-1}z^n$, and $m$ is an integer such that $\alpha + 1/p - m < 0$. It is easily verified that $f(z)(1-z)^m$ is a polynomial. Therefore

$$M_p^p(\rho, f) \le C\int_{|z|=1} |1-\rho z|^{-pm}|dz|, \quad 0 < \rho < 1.$$

If we put $z = (\zeta + \rho)/(1 + \rho\zeta)$ we see that the last integral equals

$$(1-\rho^2)^{1-pm}\int_{|\zeta|=1} |1+\rho\zeta|^{pm-2}|d\zeta|.$$

Since $pm - 2 > -1$ we find $M_p(\rho, f) \le C(1-\rho)^{1/pm}$, $0 < \rho < 1$, whence

$$\begin{aligned}\|f_r\|_{H(p,\infty,\alpha)} &\le C\sup_\rho(1-\rho)^\alpha(1-\rho r)^{1/p-m}\\ &\le C\sup -\rho(1-\rho r)^\alpha(1-\rho r)^{1/p-m}\\ &\le C(1-r)^{\alpha+1/p-m}, \quad 0 < r < 1.\end{aligned}$$

It follows that

$$(1-r)^{\alpha}M_{p_0}(r^2, f*g) \le \|g*f_r\|_{H(p_0,\infty,\alpha)} \le C(1-r)^{\alpha+1/p-m},$$

i.e. $D^{m-1}g = f*g \in H(p_0,\infty,m-1/p)$. Applying Theorem HLF, quoted in Introduction) we conclude that $D^{1/p}g \in H(p_0,\infty,1)$.

To continue the proof we need the following lemma,

LEMMA 4.1 [13]. *Let $f \in H^p$, $0<p\le 1$, and $g \in H_q$, $q \ge p$. Then*

$$M_q(r, f*g) \le (1-r)^{1-1/p}\|f\|_p\|g\|_q, \quad 0<r<1.$$

We return to the proof of Theorem 4.1. Let $D^{1/p}g \in H(p_0,\infty,1)$ and $f \in H(p,\infty,\alpha)$, $p \le 1$, $p_0 \ge p$. We have to prove that $h{:=}\, f*g$ belongs to $H(p_0,\infty,\alpha)$. We have, by the lemma,

$$M_{p_0}(r^3, D^{1/p}h) = M_{p_0}(r, f_r * D^{1/p}g_r) \le (1-r)^{1-1/p}\|f_r\|_p\|D^{1/p}g_r\|_{p_0}.$$

It follows from the hypotheses that $\|f_r\|_p \le C(1-r)^{-\alpha}$ and $\|D^{1/p}g_r\|_{p_0} \le C(1-r)^{-1}$, so that $M_{p_0}(r^3, D^{1/p}h) \le C(1-r)^{-\alpha-1/p}$, i.e. $D^{1/p}h \in H(p_0,\infty,a+1/p)$. Hence $h \in H(p_0,\infty,1)$, by Theorem HLF.

The preceding discussion shows that $D^{1/p}$ is an isomorphism of the space $(H(p,\infty,\alpha) \to H(p_0,\infty,\alpha))$ onto $H(p,\infty,1)$. By using (4.1) we conclude that, if $p, p_0 > 1/(N+1)$, then $D^{1/p}$ is an isomorphism of $(H(p,q,\alpha) \to H(p_0,q_0,\alpha))$ onto $H(p_0,\infty,1)[q_1,W]$. But the last space is equal to $H(p_0,q_1,1)$, by Theorem 4.2. This completes the proof of Theorem 4.1.

## 5. Multipliers into $l^p$ spaces

A complex sequence $\{a_n\}_{n=0}^{\infty}$ is of class $l(p,q)$ $(0<p,q\le\infty)$ if

$$\left\{\Big(\sum_{J_n}|a_j|^p\Big)^{1/o}\right\}_{n=0}^{\infty} \in l^q,$$

where $J_0=\{0\}$ and $J_n=\{j : 2^{n-1}\le j<2^n\}$, $n\ge 1$. It is easily checked that if $\{a_n\}$ is in $l(p,q)$ then the function $f(z)=\sum_0^\infty a_n z^n$ in analytic in the unit disc. Therefore $l(p,q)$ may be treated as a space of analytic functions. Furthermore, $l(p,q)$ is an $A$-space (with the obvious quasi-norm). Note that $l^p = l(p,p)$.

Let $N$ and $W$ be as in Section 4. Then we have $\|w_n * f\|_x \le C\|f\|_x$, where $X = H(\infty,q,\alpha)$ and $C$ is independent of $f, n$. In particular, taking $f(z)=z^j$ we see that $\hat{w}_n(j)| \le C$, $j,n\ge 0$. Using this one can easily prove the following.

LEMMA 5.1. *$l(p,q) = l^p[g,W]$ for all $p,q>0$.*

Now we can. use Theorem 2.1 and 4.2 to obtain

$$(H(p,q,\alpha) \to l(p_0,q_0)) = (H(p,\infty,\alpha) \to l^{p_0})[q_1,W],$$

where $p > 1/(N+1)$. If $p \leq 1$ then the space $H(p, \infty, \alpha) \to l^{p_0})$ is easily determined and is isomorphic to $l^{p_0}$, via the operator $D^{\alpha+1/p-1}$. See [12]. In the special case $p_0 = q_0 = s$ we have the following result.

THEOREM 5.1. *Let* $p \leq 1$ *and* $0 < s \leq \infty$. *A function* $g \in H(U)$ *is a multiplier from* $H(p, q, \alpha)$ *to* $l^s$ *if and only if*

$$\{(n+1)^{\alpha+1/p-1}\hat{g}(n)\}_{n=0}^{\infty} \in l(s, q_1),$$

*where* $q_1 = \infty$ *if* $s \geq q$; $q_1 = qs/(q-s)$ *if* $s < q$.

Some special cases of this theorem were proved by Ahern and Jevtić [1] ($p = 1, s > 1$) and Mateljević and Pavlović [12] ($p < 1, s \geq q$).

## 6. Problems

Let $\varphi$ be a normal function and $\alpha > 0$.

*Problem* 1. Find a function $g \in H(U)$ (if it exists) such that, for all $p, q$, the map $f \mapsto f * g$ is an isomorphism from $H(p, q, \varphi)$ onto $H(p, q, \alpha)$.

Note that the form of $g$ should be independent of $p > 0$.

By using the complex maximal theorem and our results this problem is easily reduced to the following.

*Problem* 2. Does there exist an equivalent function $\psi$ such that

$$\psi(1/t) = \int_0^1 r^t d\eta(r) \quad \text{and} \quad t^{-m}/\psi(1/t) = \int_0^1 r^t d\mu(r), \quad t > 1,$$

for some positive Borel measures $\eta$, $\mu$ and some integer $m > 0$?

*Added in proof*: The author solved Problem 1 above.

## REFERENCES


[1] P. Ahern and M. Jevtić, *Duality and multipliers for mixed norm spaces*, Michigan Math. J. **30** (1983), 53–64.

[2] G. Bennet, D.A. Stegenga and R.M. Timoney, *Coefficients of Bloch and Lipschitz functions*, Illinois J. Math. **25** (1981), 520–531.

[3] P.L. Duren, *Theory of Hp spaces*, Academic Press, 1970.

[4] P.L. Duren and A.L. Shields, *Coefficient multipliers of $H^p$ and $B^p$ spaces*, Pacific J. Math. **32** (1970), 69–78.

[5] T.M. Flett, *Lipschitz spaces of functions on the circle and the disc*, J. Math. Anal. Appl. **39** (1970) 125–158.

[6] T.M. Flett, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, ibid. **38** (1972), 746–765.

[7] G.H. Hardy and J.E. Littlewood, *Some properties of conjugate functions*, J. für Math. **167** (1931) 405–423.

[8] G.H. Hardy and J.E. Littlewood, *Some properties of fractional integrals. II*, Math. Z. **34** (1932), 403–439.

[9] G.H. Hardy and J.E. Littlewood, *Theorems concerning mean values of analytic and harmonic functions*, Quart. J. Math. **12** (1941), 221–256.

[10] C.N. Kellogg, *An extension of the Hausdorff-Yong theorem*, Mich. Math. J. **18** (1971), 121–127.

[11] M. Mateljević and M. Pavlović, *$L^p$-behaviour of the integral means of analytic functions*, Studia Math. **77** (1984), 219–237.

[12] M. Mateljević and M. Pavlović, *Duality and multipliers in Lipschitz spaces*, Proc. Internat Conf. Complex Anal. Appl. '83 (Sofia, 1985).

[13] M. Pavlović. *An inequality for the integral means of an Hadamard product* (to appear).

[14] M. Pavlović, *Mixed norm spaces of analytic and harmonic functions*, I, Publ. Inst. Math. (Beograd) **40** (**54**) (1986).

[15] A.L. Shields and D.L. Williams, *Bounded projections duality and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc. **162** (1971), 287–302.

[16] A.L. Shields and D.L. Williams, *Bounded projections duality and multipliers in spaces of harmonic functions*, J. Reine Angew. Math. **299/300** (1978), 256–279.

Odsek za matematiku
Prirodno-matematički fakultet
34000 Kragujevac
Jugoslavija

(Received 19 05 1986)