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ON EXTENSIONS OF MULTIVALUED MAPPINGS OF TOPOLOGICAL SPACES

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Abstract. For some classes of multivalued mappings $F : X \to Y$ we consider the uniqueness and existence of extensions $\overline{F} : X^* \to Y^*$, with X and Y dense in the spaces X^* and Y^* respectively.

Introduction. We assume that all mappings are pointwise-closed, i.e. each point of the domain is mapped into nonempty closed subset of the range. Note that for the multivalued mapping $F : X \to Y$, with $FX = Y$, the inverse mapping $F' : Y \to X$ is defined with $F'y = \{x | yinFx, x \in X\} \neq \emptyset$, for each $y \in Y$. If F is a multivalued mapping and $A \subset X$, the set $FA := \bigcup\{Fx | x \in A\} = \{y | F'y \cap A \neq \emptyset\} \subset Y$ is called the *image* and the set $F^+A := \{y | F'y \subset A\} \subset Y$ the *small image* of the set A by the mapping F. For any $A \subset X, F^+A \subset F'$ holds. If $B \subset Y$, the set $F'B := \bigcup\{F'y | y \in B\} = \{x | Fx \cap B \neq \emptyset\} \subset X$ is the *inverse image* and the set $F^\circ B := \{x | Fx \subset B\} \subset X$, is the *small inverse image* of the set B by the mapping F. For any $B \subset Y$, $F^\circ B \subset F'B$ holds. In addition we quote some well known relations, inclusions, implications and definitions concerning multivalued mappings, and use them subsquently.

(i) For any $A \subset X$, $F^+A = Y \setminus F(X \setminus A)$ and $FA = Y \setminus F^+(X \setminus A)$, and for any $B \subset Y$, $F^\circ B = X \setminus F'(Y \setminus B)$ and $F'B = X \setminus F^\circ(Y \setminus B)$.

(ii) For any $A \subset X$,

$$F^\circ F^+ A \subset F'F^+A \subset A \subset F^\circ FA \subset F'FA, \tag{α}$$

and for any $B \subset Y$,

$$F^+F^\circ B \subset FF^\circ B \subset B \subset F^+F'B \subset FF'B. \tag{β}$$

(iii) For all A, $A' \subset X$

$$A \cap A' = \emptyset \Rightarrow FA \cap F^+A' = \emptyset, \tag{γ}$$

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and for all B, $B' \subset Y$ $B \cap B' = \emptyset \Rightarrow F'B \cap F^\circ B' = \emptyset$. (δ)

(iv) A multivalued mapping F is *upper semicontinuous* (u.s.c.) iff the inverse mapping F' is closed ($\equiv F^\circ$ is open) and a mapping F is *lower semicontinuous* (l.s.c.) iff the inverse mapping F' is open ($\equiv F^\circ$ is closed). A multivalued mapping F is *continuous* iff F is upper and lower semicontinuous. A multivalued mapping F is Y*-compact* iff the image Fx is compact, for every $x \in X$ and a mapping F is X*-compact* iff the inverse image $F'y$ is compact, for each $y \in Y$.

Some criterions of semicontinuity. The next theorem gives some criterions of semicontinuity of multivalued mappings.

THEOREM 1. *Let $F : X \to Y$ be a multivalued mapping.*

(i) *A mapping F is u.s.c. iff for any closed $B \subset Y$ and any other $B' \subset Y$*

$$B \cap B' = \emptyset \Rightarrow \overline{F'B} \cap F^\circ B' = \emptyset \tag{1}$$

(ii) *A mapping E is l.s.c. iff for any closed $B \subset Y$ and any other $B' \subset Y$,*

$$B \cap B' = \emptyset \Rightarrow \overline{F^\circ B} \cap F^\circ B' = \emptyset. \tag{2}$$

Proof. (a) Let $B \subset Y$ be closed and $B' \subset Y$ any other set such that $B \cap B' = \emptyset$. Then $F'B \cap F^\circ B' = \emptyset$ follows from (δ). If F is u.s.c. mapping, then $F'B = \overline{F'B}$ since F' is closed. Thus $\overline{F'B} \cap F^\circ B' = \emptyset$ and (1) is proved. If F is l.s.c. mapping, then $F^\circ B = \overline{F^\circ B}$ since F° is closed. Thus $\overline{F^\circ B} \cap F'B' = \emptyset$ so (2) is proved.

(b) Let now the implications (1) and (2) hold, and let $B \subset Y$ be a closed set. If we put $B' = Y \setminus B$, then $B \cap B' = \emptyset$. By (1) we have $\overline{F'B} \cap F^\circ B' = \emptyset$ and further $\overline{F'B} \subset X \setminus F^\circ B' = X \setminus F^\circ(Y \setminus B) = F'B$, i.e. $\overline{F'B} \subset F'B$. So we have proved that the mapping F' is closed and that F is u.s.c. Also $B \cap B' = \emptyset$ implies $\overline{F^\circ B} \cap F'B' = \emptyset$ by (2) and further $\overline{F^\circ B} \subset X \setminus F'(Y \setminus B) = F^\circ B$, i.e. $\overline{F^\circ B} \subset F^\circ B$. Thus we have proved that the mapping F° is closed and that F is l.s.c.

COROLLARY 1.1. *Let $f : X \to Y$ be a single valued mapping. Then f is continuous iff for any closed set $B \subset Y$ any $B' \subset Y$ we have*

$$B \cap B' = \emptyset \Rightarrow \overline{f^{-1}B} \cap B' = \emptyset.$$

Corollary 1.1. follows from Theorem 1 since any single valued mapping f is continuous, if it is u.s.c. or l.s.c. and is $f^{-1} = f^\circ$.

Results. When we discuss the extension of multivalued mapping there naturally appears the question of uniqueness of these extensions. In connection with this is the following

THEOREM 2. *Let $F : X \to Y$ be a multivalued mapping and $X*, Y*$ spaces such that $\overline{X} = X^*$, $\overline{Y} = Y^*$.*

(i) *If the space Y^* is a T_3-space and $\overline{F} : X^* \to Y^*$ is a l.s.c. extension, $\overline{F}_1 : X^* \to Y^*$ is u.s.c. extension, or*

(ii) *if the space Y^* is a T_2-space and $\overline{F}: X^* \to Y^*$ is l.s.c. extension, $\overline{F}_1 : X^*Y^*$ is u.s.c. and Y^*-compact extension of the mapping F, then $\overline{F}x^* \subset \overline{F}_1x^*$ for each $x^* \in X^*$.*

Proof. If there is a point $x^* \in X^*$, such that $\overline{F}x^* \setminus \overline{F}_1x^* \neq \emptyset$, let $y \in \overline{F}x^* \setminus \overline{F}_1x^*$. Since the space Y is T_3-space, or T_2-space and $\overline{F}_1x^*$ a compact subspace and $y \notin \overline{F}_1x^*$, there are open sets V^* and W^* in a $Y*$, such that $y \in V^*$, $\overline{F}_1x^* \subset W^*$ and $V^* \cap W^* = \emptyset$. From $y \in \overline{F}x^*$ and $y \in V^*$ it follows $\overline{F}x \cap V^* \neq \emptyset$ and $x^* \in \overline{F}'V^* = U_0^*$. The set $U_0^* = \overline{F}'V^*$ is open, since the mapping F is l.s.c. From $\overline{F}_1x^* \subset W^*$ it follows $x^* \in \overline{F}_1^\circ W^* = U_1^*$ and the set U_1^* is open too, since the mapping $\overline{F}$ is u.s.c. Then the set $U^* = U_0^* \cap U_1^*$ is open in X^* and $U^* \neq \emptyset$, because $x^* \in U^*$. Since the set X is dense in the space X^*, we have. $U^* \cap X \neq \emptyset$. If $x \in U^* \cap X$, then $x \in U_0^* = \overline{F}'V^*$ and $x \in U_1^\circ = U_1^\circ W^*$. But the mappings $\overline{F}$ and $\overline{F}_1$ are extensions of the mapping F and $Fx = \overline{F}x = \overline{F}_1x$, if $x \in X$. Then from $x \in \overline{F}'V^*$ follows $\emptyset \neq \overline{F}x \cap V^* = Fx \cap V^*$ and from $x \in \overline{F}_1^\circ W^*$ follows $Fx = \overline{F}_1x \subset W^*$. Thus $\emptyset \neq Fx \cap V^* \subset W^* \cap V^*$, i.e. $V^* \cap W^* \neq \emptyset$ and we have the contradiction with $V^* \cap W^* = \emptyset$. Hence $\overline{F}x^* \subset \overline{F}_1x^*$, for each $x^* \in X^*$.

COROLLARY 2.1. *Let $\overline{F} : X^* \to Y^*$ be an extension of a multivalued mapping $F : X \to Y$.*

(i) *If the space Y^* is T_2-space and the extensions $\overline{F}$ continuous, or*

(ii) *if the space Y^* T_2-space und the mapping $\overline{F}$ continuous and Y^*-compact, then $\overline{F}$ is the unique extension of the mapping F.*

Proof. If there are two continuous ($\equiv$ u.s.c. and l.s.c.) extensions $\overline{F}$ and $\overline{F}_1$ of the mapping F, then for each $x^* \in X^*$, $\overline{F}x^* \subset \overline{F}_1x^*$, since $\overline{F}$ is l.s.c. and $\overline{F}_1$ u.s.c. mappings, and for each $x^* \in X^*$, $\overline{F}_1x^* \subset \overline{F}x^*$, since $\overline{F}_1$ is l.s.c. and $\overline{F}$ u.s.c. mapping. So we have $\overline{F}x^* = F_1x^*$, for each $x^* \in X^*$, i.e. $\overline{F} = \overline{F}_1$. Thus, with the conditions (i) or (ii) there is the unique extension $\overline{F}$ of F.

We shall now consider the existence of an extension $\overline{F}$ of a multivalued mapping F. It is known that for a single valued mapping $f : X \to Y$, if $Y = Y^*$ there is not a continuous extension $\overline{f} : X^* \to Y$, where $X \subset X^*$ and $\overline{X} = X^*$ iff the mapping f is perfect, [4]. A similar assertion for the multivalued mappings is not true, because it can be proved that there are several classes of multivalued mappings that cannot be extended as u.s.c. or l.s.c. mappings. Such classes of mappings contain necessarily the class of perfect single valued mappings. What is the largest class of multivalued mappings which cannot he extended as u.s.c. or l.s.c. mapping is an open question.

THEOREM 3. *X^* and $Y = Y^*$ be T_2-spaces and $F : X \to Y$ a multivalued mapping, where $\overline{X} = X^*$. Then the mapping F can not be extended to the whole X^** (i) *as u.s.c. and Y-compact mapping, if the mapping F is perfect (u.s.c., closed, X-compact and Y-compact), or* (ii) *as l.s.c. mapping, if F is l.s.c., closed and X-compact mapping.*

Proof. (i) If there was an extension $F : X^* \to Y$ of the mapping F which is u.s.c. and Y-compact, then for each point $x^* \in X^* \setminus X$, the set $\overline{F}x^*$ is compact in Y. Then the set $F'\overline{F}x^* \subset X$ compact in X as in X^*, since the inverse mapping $F' : Y \to X$ perfect as the mapping F is perfect. Because $x^* \in X^* \setminus X \subset X^* \setminus F'\overline{F}x^*$ and set $F'\overline{F}x^*$ is compact in T_2-space X^*, there are two open in X^* sets O^* and U^*, such $x^* \in O^*$, $F'\overline{F}x^* \subset U^*$ and $O^* \cap U^* = \emptyset$. Notice that the set $V = F^+U^* = \{y | F'y \subset U^*\} = \{y | F'y \subset U^* \cap X\} = F^+(U^* \cap X)$ is open in Y, since the mapping F is closed and the set $U^* \cap X \neq \emptyset$ is open in X. From (β) follows $\overline{F}x* \subset F^+F'(\overline{F}x^*) \subset F^+U^* = V$, i.e. $\overline{F}x^* \subset V$. Since the mapping $\overline{F}$ is u.s.c. and the set V is open in Y, the set $O_1^* = \overline{F}^\circ V$ is open in $X^* \cdot \overline{F}x^* \subset V$ implies $x^* \in F^\circ V = O_1^*$. The set $O_0^* = O^* \cap O_1^*$ is open and $O_0^* \neq \emptyset$, since $x \in O_0^*$. Since X is dense in the space X^*, $O_0^* \cap X \neq \emptyset$. If $x \in O_0^* \cap X$, then $x \in O^*$ and $x \in O_1^*$. Then $x \in O_1^* = \overline{F}^\circ V$ implies $Fx = \overline{F}x \subset V = F^+U^*$, and further $x \in F^\circ V \subset F^\circ F^+U^* = F^\circ F^+(U^* \cap X) \subset U^* \cap X \subset U^*$, i.e. $x \in U^*$. Thus $x \in U^*$ and $x \in O^*$ implies $O^* \cap U^* \neq \emptyset$ and we get the contradiction with $O^* \cap U^* = \emptyset$. So (i) must be true.

(ii) Let $x^* \in X^* \setminus X$ and $y \in \overline{F}x^*$ be arbitrary points. Since the mapping F is X-compact, the set $F'y \subset X$ is compact in X as in X^*. As the space X^* is a T_2-space there are two open in X^* sets O^* and U^*, such that $x^* \in O^*$, $F'y \subset U^*$ and $O^* \cap U^* = \emptyset$. By $F'y \subset U^*$ it follows that $y \in F^+U^a st = F^+(U^* \cap X) = V$ and the set V is open in Y since the set $U^* \cap X \neq \emptyset$ is open in X and the mapping F is closed ($\equiv F^+$ -open). From $y \in \overline{F}x^*$ and $y \in V$ it follows that $\overline{F}x^* \cap V \neq \emptyset$ and $x^*\overline{F}'V = O_1^*$. The set O_1^* is open in X^*, since the extension $\overline{F}$ is l.s.c. Thus $x^* \in O^* \cap O_1^* = O_0^* \neq \emptyset$ and $O_0^* \cap X \neq \emptyset$, since O_0^* is open in X^* and the set X dense in X^*. If $x \in O_0^* \cap X$, then $x \in O^*$ and $x \in O_1^*$. Since $x \in X$, $Fx = \overline{F}x$. From $x \in 0_1^* = \overline{F}'V$ follows $Fx \cap V = \overline{F}x \cap V \neq \emptyset$. If $y \in Fx \cap V$, then $y \in Fx$ and $y \in V = F^+U^*$. This implies that $x \in F'y \subset U^*$, i.e. $x \in U^*$ and since $x \in O^*$ we have $O^* \cap U^* \neq \emptyset$ which contradicts $O^* \cap U^* = \emptyset$. Thus (ii) must be true and the the theorem is proved.

COROLLARY 3.1. *If a multivalued mapping $F : X \to Y$ is* (i) *perfect and there is an extension $\overline{F} : X^* \to Y^*$ which is u.s.c. and Y compact, or* (ii) *l.s.c., closed and X compact and there is an extension $\overline{F} : X^* \to Y^*$ which is l.s.c. and the spaces are T_2-spaces, then $Y^* \setminus Y \neq \emptyset$.*

COROLLARY 3.2. *If a multivalued mapping $F : X \to Y$ has an extension $\overline{F} : X^* \to Y^*$ under conditions* (i) *or* (ii) *of Theorem 3 then $\overline{F}'(Y^* \setminus Y) = X^* \setminus X$ and $\overline{F}^\circ Y = X$.*

Proof. If the extension $\overline{F}$ of F satisfies one of the conditions (i) or (ii), then $\overline{F}x^* \cap (Y^* \setminus Y) \neq \emptyset$, for each $x^* X^* \setminus X$. Hence $x^* \in \overline{F}'(Y^* \setminus Y)$ and the implication $x^* X^* \setminus X \Rightarrow x^* \in \overline{F}'(Y^* \setminus Y)$ is proved, from which follows the inclusion $(X^* \setminus X) \subset \overline{F}'(Y^* \setminus Y)$. From $(Y^* \setminus Y) \cap Y = \emptyset$ and (δ) follows $\overline{F}'(Y^* \setminus Y) \cap \overline{F}^\circ Y = \emptyset$ and since $Fx = \overline{F}x \subset Y$, for each $x \in X$ we have $X \subset \overline{F}^\circ Y$, and further

$\overline{F}'(Y^* \setminus Y) \cap X = \emptyset$. Hence $\overline{F}'(Y^* \setminus Y) \cap X^* \setminus X$ and the first equality is proved. By $\overline{F}'(Y^* \setminus Y) \cap \overline{F}^\circ Y = \emptyset$, the first equality implies $(X^* \setminus X) \cap \overline{F}^\circ Y = emptyset$ and also $\overline{F}^\circ Y \subset X^* \setminus (X^* \setminus X) = X$. As $X \subset \overline{F}^\circ Y$ the second equality is proved.

The main result in this paper is the following criterion of existence of an extension of a multivalued continuous mapping, if $Y = Y^*$ is compact space.

THEOREM 4. *Let X he a dense set in a T_2-space X^* and $F : X \to Y$ a multivalued continuous mapping into a T_2-space Y. Then there is a unique continuos extension $\overline{F} : X^* \to Y$ of the mapping F iff for each pair $B, B' \subset Y$ of closed sets*

$$B \cap B' = \emptyset \qquad \overline{F'B}^* \cap \overline{F^\circ B'}^* = \emptyset, \tag{1}$$

where $\overline{F'B}^*$ and $\overline{F^\circ B'}^*$ are the closures of sets in the space X^*.

Proof. (a) Let a mapping F has a continuous extension $F : X^* \to Y$ and $B, B' \subset Y$ be nonempty, disjoint and closed in Y. Then $\overline{F}'B \cap \overline{F}^\circ B' = \emptyset$ by (δ). Since $\overline{F}$ is u.s.c. and l.s.c., the sets $\overline{F}'B$ and $\overline{F}^\circ B \setminus X^* \setminus \overline{F}'(Y \setminus B')$ are closed in X^*. By $F'B \subset \overline{F}'B$ and $F^\circ B' \subset B'$ we get $\overline{F'B}^* \subset \overline{\overline{F}'B}^* = \overline{F}'B$ and $\overline{F^\circ B'}^* \subset \overline{F^\circ B'}^* = \overline{F}^\circ B'$. Thus from $\overline{F}'B \cap \overline{F}^\circ B' = \emptyset$. follows $\overline{\overline{F}'B}^* \cap \overline{F^\circ B'}^* = \emptyset$ and the implication (1) is proved.

(b) Let now the implication (1) holds for the mapping F and any pair of closed, disjoint sets $B, B' \subset Y$. Denote by $\mathcal{U}(x^*)$ the family of all open in X^* neighborhoods U^* of a point $x^* \in X^*$. The extension $\overline{F} : X^* \to Y$ of the mapping F we define by the following equality

$$\overline{F}x := \bigcap \{\overline{F(U^* \cap X)} | \, U^* \in \mathcal{U}(x^*)\}, \tag{2}$$

for point $x^* \in X^*$.

1. Since the set X is dense in the space X^*, then $U^* \cap X \neq \emptyset$ for any set $U^* \in \mathcal{U}(x^*)$ and therefore $\overline{F(U^* \cap X)} \neq \emptyset$, for any set $U^* \in \mathcal{U}(x^*)$. So the family $\mathcal{F}(x^*) = \{\overline{F(U^* \cap X)} | \, U^* \in \mathcal{U}(x^*)\}$ is the family of nonempty, closed in Y sets. If $U_1^*, U_2^*, \ldots, U_n^* \in \mathcal{U}(x^*)$, then $U_0^* = \bigcap_{i=1}^n \bigcup_i^* \in \mathcal{U}(x^*)$, and $\emptyset \neq \overline{F(U_0^* \cap X)} \subset \overline{F(U_i^* \cap X)}$, for $i = 1, 2, \ldots, n$. Thus $\emptyset \neq \overline{F(U_0^* \cap X)} \subset \bigcap_{i=1}^n \overline{F(U_i^* \cap X)}$ and it is proved that the family $\mathcal{F}(x^*)$ has the finite intersection property. Since the space Y is compact, then $\bigcap \{\overline{F(U^* \cap X)} | \, U^* \in \mathcal{U}(x^*) \neq \emptyset$, i.e. $\overline{F}x \neq \emptyset$, for each $x^* \in X^*$.

2. If $x \in X$ then $x \in U^* \cap X$ and $Fx \subset \overline{F(U^* \cap X)}$ for any set $U^* \in \mathcal{U}(x)$. Thns $Fx \subset \bigcap \{\overline{F(U^* \cap X)} | \, U^* \in \mathcal{U}(x^*)\} = \overline{F}x$, i.e. $Fx \subset \overline{F}x$. We shall prove that $\overline{F}x \setminus Fx \neq \emptyset$ is impossible. If $y \in \overline{F}x \setminus Fx$, as the space Y is a T_2-compact space, there are two open in Y sets V and V', such that $y \in V$, $Fx \subset V'$ and $V \cap V' = \emptyset$. By the implication (1) we obtain now $\overline{F'\overline{V}}^* \cap \overline{F^\circ \overline{V'}}^* = \emptyset$. As $Fx \subset V'$ iff $x \in F^\circ V' \subset \overline{F^\circ \overline{V'}}$, we have $x \notin \overline{F'\overline{V'}}^*$, i.e. $x \in X^* \setminus \overline{F^\circ \overline{V}}^* = U_0^*$ and $U_0^* \mathcal{U}(x)$. Then $U_0^* \cap \overline{F'\overline{V}}^* = \emptyset$ and also $U_0^* F'V = \emptyset$. Therefore $(U_0^* \cap X) \cap F'V = \emptyset$ and by

(γ) we have $F(U_0^*\cap X)\cap F^+F'V=\emptyset$ or $F(U_0^*\cap X)\cap V=\emptyset$ since $V\subset F^+F'V$. As the set V is open Y, it must be $\overline{F(U_0^*\cap X)}\cap V=\emptyset$ and also $\overline{F}x\cap V=\emptyset$ since $\overline{F}x=\bigcap\{\overline{F(U^*\cap X)}|\,U^*\in\mathcal{U}(x)\}\subset\overline{F(U_0^*\cap X)}$. On the other hand, from $y\in\overline{F}x$ and $y\in V$ follows $\overline{F}x\cap V\neq\emptyset$ and we have a contradiction. Hence $\overline{F}x\setminus Fx\neq\emptyset$ can not hold and $Fx=\overline{F}x$, if $x\in X$. So we proved that the mapping $\overline{F}$ defined by equality (2) is an extension of the mapping F.

3. The continuity of the extension F will be proved using Theorem 1. Therefore we shall prove two following inclusions: $\overline{F}'B\subset\overline{F'V}^*$ and $\overline{F}^\circ B\subset\overline{F^\circ V}^*$, where B is closed and V open in Y set such that $B\subset V$.

(i) If $x^*\overline{F}\in B$, then $\overline{F}x^*\cap B\neq\emptyset$ and therefore $\overline{F(U^*\cap X)}\cap B\neq\emptyset$, for each $U^*\in\mathcal{U}(x^*)$. Then $\overline{F(U^*\cap X)}\cap V\neq\emptyset$ and further $F(U^*\cap X)\cap V\neq\emptyset$, since the set V is open in Y and $B\subset V$. Hence there is a point $x\in U^*\cap X$ such that $Fx\cap V\neq\emptyset$ and $x\in(U^*\cap X)\cap F'V\subset U^*\cap F'V$. Thus $U^*\cap F'V\neq\emptyset$, for each set $U^*\in\mathcal{U}(x^*)$ and $x^*F'V^*$. So we proved the implication $x^*\in\overline{F}'B\Rightarrow x\in\overline{F'V}^*$, which gives the inclusion $\overline{F}'B\subset\overline{F'V}^*$.

(ii) If $x^*\in\overline{F}^\circ B$, then $\overline{F}x^*\subset B$ and $\overline{F}x^*\subset V$, since $B\subset V$. We shall prove that there is a neighborhood $U_0^*\in\mathcal{U}(x^*)$ such that $\overline{F(U_0^*\cap X)}\subset V$. In fact, if for any neighborhood $U^*\subset\mathcal{U}(x^*)$ we suppose that $\overline{F(U^*\cap X)}\setminus V\neq\emptyset$ then the family $\{\overline{F(U^*\cap X)}\setminus V|\,U^*\in\mathcal{U}(x^*)\}$ is the family of closed nonempty sets in Y. It is easy to show that this family has the finite intersection property and, since the space Y is compact, is must be

$$\emptyset\neq\bigcap\{\overline{F(U^*X)}|\,U^*\in\mathcal{U}(x^*)\}=\bigcap\{\overline{F(U^*X)}|\,U\in\mathcal{U}(x^*)\}\setminus V=\overline{F}x^*\setminus V,$$

i.e. $\overline{F}x^*\setminus V\neq\emptyset$ and we have the contradiction with $\overline{F}x^*\subset V$. Thus there is a neighborhood $U_0^*\in\mathcal{U}(x^*)$ such that $\overline{F(U_0^*\cap X)}\subset V$. If $U^*\in\mathcal{U}(x^*)$, then the set $U^*\cap U_0^*$ is a neighborhood of the point x^* and $F((U^*\cap U_0^*)\cap X)\subset F(U_0^*\cap X)\subset\overline{F(U_0^*\cap X)}\subset V$. So we have $\emptyset(U_0^*U^*)\cap X\subset F^\circ V$ and $U^*\cap F^\circ V\neq\emptyset$, for each neighbourhood $U\in\mathcal{U}(x^*)$ and therefore $x^ast\in\overline{F^\circ V}^*$. So we proved the implication $x^*\in\overline{F}^\circ B\Rightarrow x^*\in\overline{F^\circ V}^*$, i.e. the inclusion $\overline{F}^\circ B\subset\overline{F^\circ V}^*$.

Let now $B,B\subset Y$ be any pair of closed, nonempty and disjoint sets. Since each T_2-compact space is T_4-space, there are two open sets $V,V'\subset Y$ such that $B\subset V$, $B'\subset V'$ and $\overline{V}\cap\overline{V}'=\emptyset$. By implication (1) then follows $\overline{F'\overline{V}}^*\cap F^\circ\overline{V}'=\emptyset$. As $\overline{F}'B\subset\overline{F'V}^*$ implies $\overline{\overline{F}'B}^*\subset\overline{F'V}^*$ and $\overline{F^circB'}^*\subset\overline{F^\circ V}^*$ implies $\overline{\overline{F}^\circ B'}^*\subset\overline{F^\circ V'}^*$ and $B\cap B'=\emptyset$ we have two implications

$$B\cap B'=\emptyset\Rightarrow\overline{\overline{F}'B}^*\cap\overline{F}^\circ B'=\emptyset,\tag{3}$$

$$B\cap B'=\emptyset\Rightarrow\overline{F}'B\cap\overline{\overline{F}^\circ B'}^*=\emptyset.\tag{4}$$

By Theorem 1 and (3) it follows that the extension $\overline{F}$ is u.s.c. and by (4) that the extension $\overline{F}$ is l.s.c., i.e. the extension $\overline{F}$ is continuous. Corollary 2.1 implies that the extension $\overline{F}$ is unique and the theorem is proved.

COROLLARY 4.1. *Let X be a dense set in a T_2-space X^* and $f : X \to Y$ a continuous single valued mapping into a T_2-compact space Y. Then there is an extension $f : X^* \to Y$ of the mapping f iff for each pair of closed sets $B, B' \subset Y$,*

$$B \cap B' = \emptyset \Rightarrow \overline{f^{-1}B}^* \cap \overline{f^{-1}B'}^* = \emptyset$$

The proof of this corollary follows from Theorem 4 since $F = f^{-1}$ if $F = f$ is a single valued mapping [4, Theorem 3.2.1]. Using now the preceding theorem we can prove the existence of extensions of Y-compact multivalued mapping of a normal space.

THEOREM 5. *For every T_2-compactification $\alpha\ Y$ of a Tychonoff space Y and every multivalued continuous and Y compact mapping $F : X \to Y$ of a normal space X into a space Y, there is a continuous extension $\overline{F} : \beta X \to \alpha Y$, where βX is the Stone-Čech compactifaction of the space X.*

Proof. Let αY be any T_2-compactification of the space Y. Since the mapping F is Y-compact every set Fx is compact in the space Y as in the space αY and therefore is closed in αY, if $x \in X$. If we define a multivalued mapping $F_\alpha : X \to \alpha Y$ by $F_\alpha x = Fx$, for each $x \in X$, then it is a pointwise-closed mapping. We need to prove that the mapping F_α is continuous ($\equiv$ u.s.c. and l.s.c.) and that the condition (1) for the mapping F_α holds.

Since $F_\alpha x = Fx \subset Y$ for each $x \in X$, then for any closed in αY set B^* holds $F'_\alpha B^* = F'(B^* \cap Y)$ and $F^\circ_\alpha B = F^\circ(B^* \cap Y)$. Since the set $B^a st \cap Y$ is closed in Y and the mapping F is u.s.c. and l.s.c., the sets $F'_\alpha B^* = F' \cap Y$ and $F^\circ_\alpha B^* = F^\circ(B^* \cap Y)$ are closed in the space X. Thus the mapping F_α is u.s.c. and l.s.c., i.e. F_α is continuous.

Let $B^*, B_1^* \subset Y$ be any pair of closed sets such that $B^* \cap B_1^* = \emptyset$. Then $F'_\alpha B^* \cap F^\circ_\alpha B_1^* = \emptyset$ by (δ). The sets $F'_\alpha B^*$ and $F - \alpha^\circ B_1^*$ are closed in the normal space X and since they are disjoint, their closures in the space βX are disjoint too, i.e. $\overline{F'_\alpha B^*}^* \cap \overline{F^\circ_\alpha B_1^*}^* = \emptyset$. Thus we have proved the implication (1) of the Theorem 4 and there is a continuous extension $\overline{F} = \overline{F}_\alpha : \beta X \to \alpha Y$ of the mapping F_α which is an extension of the mapping F too.

COROLLARY 5.1. *For every T_2-compactification $a\ Y$ of a Tychonoff space Y and every continuous multivalued mapping $F : X \to \alpha Y$ of a normal space into a space Y, there is a unique continuous mapping $\overline{F}_\alpha : \beta X \to \alpha Y$ which is an extension of the multivalued mapping $F_\alpha : \beta X \to \alpha Y$ defined by $F_\alpha x = \overline{Fx}^*$, for each $x \in X$.*

Proof. We must prove that the mapping $F_\alpha : X \to \alpha Y$ is continuous when the mapping $F : X \to Y$ is continuous. We shall prove that the sets $F^\circ_\alpha V^*$ and $F'_\alpha V^*$ are open in the space X, whenever the set $V^* \subset \alpha Y$ is open in αY.

(i) If $x \in V^*$, then $F_\alpha x \subset V^*$ and $Fx \subset \overline{Fx}^* = F_\alpha x \subset V^*$. As the space $a\ Y$ is normal there is an open in αY set V_1^* such that $Fx \subset \overline{Fx}^* = F_\alpha x \subset$

$V_1^* \subset \overline{V_1^*}^* \subset V^*$. Then by $Fx \subset V_1^*$ and $Fx \subset Y$ follows $Fx \subset V_1^* \cap Y$ and therefore $x \in F^\circ(V_1^* \cap Y) = U_x$. The set $U_x = F^\circ(V_1^* \cap Y)$ is open in X. since the mapping F is u.s.c. and the set $V_1^* \cap Y$ is open Y. Now we shall prove the inclusion $U_x \subset F_\alpha^\circ V^*$. If $x' \in U_x$, then $Fx' \subset V_1^* \cap Y \subset V_1^*$, i.e. $Fx' \subset V_1^*$ and also $F_\alpha x' = \overline{Fx'}^* \subset \overline{V_1^*}^* \subset V^*$ and $x' \in F_\alpha^\circ V^*$. So the implication $x' \in U_x \Rightarrow x' \in F_\alpha^\circ V^*$ is proved which implies $U_x \subset F_\alpha^c ircV^*$. This inclusion shows that the set $F_\alpha^\circ V^*$ is a neighborhood in X of all its points, and the set $F_\alpha^\circ V^*$ is open in X. So the mapping F_α is u.s.c.

(ii) If the set $V^* \subset \alpha Y$ is open in αY, we shall prove the equality $F'_\alpha V^* = F'(V^*(\cap Y)$. Notice that $x \in F'_\alpha V^*$ iff $F_\alpha x \cap V^* \neq \emptyset$ or $\overline{Fx}^* \cap V^* \neq \emptyset$, since $F_\alpha x = \overline{Fx}^*$. But $\overline{Fx}^* \cap V^* \neq \emptyset$ iff $Fx \cap V^* \neq \emptyset$ since the set V^* is open αY. As $Fx \subset Y$, $Fx \cap V^* \neq \emptyset$ iff $Fx \cap (V^* \cap Y) \neq \emptyset$ and $Fx \cap (V^* Y) \neq \emptyset$ iff $x \in F'(V^* \cap Y)$. So we have proved the equivalence $x \in F'_\alpha V^* \Leftrightarrow x \in F'(V^* \cap Y)$ which implies the equality $F'_\alpha V^* = F'(V^* Y)$. From this equality it follows that the mapping F_α is l.s.c., since the mapping F is l.s.c. and the set $F'(V^* \cap Y)$ is open, for each open in a Y set V^*. By Theorem 5 we get the corollary 5.1.

COROLLARY 5.2. *For every T_2-compactification a Y of a Tychonoff space Y and every continuous single valued mapping $f : X \to Y$ of normal space X into a space Y, there is a unigue continuous extension $\overline{f}_\alpha : \beta X \to \alpha Y$ defined on the whole Stone-Čech compactification βX of the space X [4, Corollary 3.6.6].*

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