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ON THE DEFINITION OF A QUADRATIC FORM

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Abstract. In the first part of this paper we give a simple proof of the following wellknown theorem [3]: If a function $q : X \to C$ satisfies the parallelogram law and the homogeneity property $q(\lambda x) = |\lambda|^2 q(x)$ $(\lambda \in C, x \in X)$, then there exists a sesquilinear form $L : X \times X \to C$ such that $q(x) = L(x; x) \quad (x \in X)$.

If X is a real vector space then a quadratic form on X is to be defined as a function $q : X \to R$ the complexification $(q_c(q_c(x+iy) = q(x)+q(y); x, y \in X)$ of which has the homogeneity property
$$q_c(\lambda z) = |\lambda|^2 q_c(z) \quad (\lambda \in C, z \in X_c = X \times X).$$

In the second part of this paper we continue the study of quadratic forms on modules over algebras studied in [6], [7] and [4]. We assume as in [4] that the algebra A has the identity element and that it as the regularity property: For any $t \in A$ there exists a natural number n such that $t + n$ and $t + n + 1$ are invertible in A.

1. On the definition of a quadratic form

If X is a complex vector space and $L : X \times X \to C$ a sesquilinear form, then a function

$$q(x) = L(x, x) \qquad (x \in X) \tag{1}$$

has properties:

$$q(x + y) + q(x - y) = 2q(x) + 2q(y) \qquad (x, y \in X), \tag{2}$$

$$q(\lambda x) = |\lambda|^2 q(x) \qquad (\lambda \in C; x \in X). \tag{3}$$

Proof. I. Halperin in 1963 (The New Scottish Book) asked whether for a function $q : X \to C$ which satisfies (2) and (3) there exists a sesquilinear form such that (1) holds. The positive answer to this question was given in [3] and the proof was simplified in [5]. Here we give even simpler proof of this fact.

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THEOREM 1 [3]. *If X is a complex vector space and a function $q : X \to C$ satisfies conditions* (2) *and* (3) *then a functional $L : X \times X \to C$ defined by*

$$L(x,y) = (q(x+y) - q(x-y))/4 + i(q(x+iy) - q(x-iy))/4 \quad (x,y \in X) \tag{4}$$

is sesquilinear and (1) holds true.

We need three lemmas for the proof of this theorem. Although lemmas 1, 2 are well-known we prove them here for the convenience of a reader.

LEMMA 1. *If a function $q : X \to C$ satisfies the parallelogram law* (2), *then a function*

$$S(x,y) = q(x+y) - q(x-y) \qquad (x,y \in X)$$

is biadditivem symmetric and $4q(x) = S(x,x) \quad (x \in X)$.

Proof. From (2) for $x = y = 0$ we get $q(0) = 0$; for $x = 0$ we get $q(-y) = q(y)$ i.e. q is an even function. By taking $x = y$ in (2) we get $q(2x) = 4q(x)$.

For $x, y, u \in X$ we have:

$$\begin{aligned}
S(x+y,2u) &= q(x+y+2u) - q(x+y-2u) \\
&= q((x+u)+(y+u)) + q((x+u)-(y+u)) \\
&\quad - q((x-u)+(y-u)) - q((x-u)-(y-u)) \\
&= (2q(x+u) + 2q(y+u)) - (2q(x-u) + 2q(y-u)) \\
&= 2S(x,u) + 2S(y,u).
\end{aligned}$$

From here for $y = 0$ and $x = z$ we get $S(z,2u) = 2S(z,u)$ which for $z = x+y$ leads to

$$S(x+y,u) = S(x,u) + S(y,u). \quad \square$$

LEMMA 1. *If $q : X \to C$ satisfies* (2) *and* (3) *then the function $L : X \times X \to C$ defined by* (4) *is biadditive, $q(x) = L(x,x) \quad (x \in X)$ and*

$$L(ix,y) = iL(x,y), \qquad L(x,iy) = -iL(x,y) \qquad (x,y \in X). \tag{6}$$

Proof. Using (4) and (3) for $\lambda = i$ we have

$$\begin{aligned}
4L(ix,y) &= (q(ix+y) - q(ix-y)) + i(q(ix+iy) - q(ix-iy)) \\
&= q(x-iy) - q(x+iy) + i(q(x+y) - q(x-y)) = 4iL(x,y), \\
4L(x,iy) &= (q(x+iy) - q(x-iy)) + i(q(x+i\cdot iy) - q(x-i\cdot iy)) \\
&= q(x+iy) - q(x-iy) + i(q(x-y) - q(x+y)) = -4iL(x,y). \quad \square
\end{aligned}$$

LEMMA 3 [5]. *If $f : C \to C$ is an additive function and*

$$f(\lambda) = |\lambda|^2 f(1/\lambda) \qquad (\lambda \in C, \lambda \neq 0)$$

then

$$f(\lambda) = f(1)\,\mathrm{Re}\,\lambda \quad (\lambda \in C).$$

Proof. A function $g(\lambda) = f(1)\,\mathrm{Re}\lambda - f(\lambda)$ is additive and

$$g(1) = 0,\ g(\lambda) = |\lambda|^2 g(1/\lambda), \quad \lambda \neq 0.$$

If $\lambda \neq 0$ then

$$g(\lambda) = g(1+\lambda) = |1+\lambda|^2 g\left(\frac{1}{1+\lambda}\right) = |1+\lambda|^2 g\left(1 - \frac{\lambda}{1+\lambda}\right) = -|1+\lambda|^2 g\left(\frac{\lambda}{1+\lambda}\right) =$$
$$-|1+\lambda|^2 \left|\frac{\lambda}{1+\lambda}\right|^2 g\left(\frac{1+\lambda}{\lambda}\right) = -|\lambda|^2 g(1+1/\lambda) = -|\lambda|^2 g(1/\lambda) = -g(\lambda).$$

Thus $g = 0$ and (8) follows. $\square$

Proof of Theorem 1. For any $x, y \in X$ define

$$f(\lambda) = L(\lambda x, y) + L(x, \lambda y) \qquad (\lambda \in C). \tag{9}$$

Obviously, $\lambda \to f(\lambda)$ is an additive function. By use of (3) for $\lambda = 0$ we have

$$\begin{aligned} 4L(\lambda x, y) &= q(\lambda x + y) - q(\lambda x - y) + i(q(\lambda x + iy) - q(\lambda x - iy)) \\ &= |\lambda|^2[(q(x + y/\lambda) - q(x - y/\lambda)) + i(q(x + y/\lambda) - q(x - iy/\lambda))] \\ &= 4|\lambda|^2 L(x, y/\lambda). \end{aligned}$$

In the same way we get $L(x, \lambda y) = |\lambda|^2 L(y/\lambda, y)$.

Thus the function (9) satisfies conditions of Lemma 3 so that $f(\lambda) = f(1)\,\mathrm{Re}\lambda$, i.e.

$$L(\lambda x, y) + L(x, \lambda y) = 2L(x, y)\,\mathrm{Re}\lambda \quad (\lambda \in C; x, y \in X). \tag{10}$$

If $\lambda = it$ ($t \in \mathbf{R}$), then (10) and Lemma 2 imply

$$L(tx, y) = L(x, ty)\ (t \in \mathbf{R};\ x, y \in X). \tag{11}$$

If $\lambda = t$ ($t \in \mathbf{R}$), then (10) implies

$$L(tx, y) + L(x, ty) = 2tL(x, y)$$

which together with (11) leads to

$$L(tx, y) = tL(x, y) \qquad (t \in \mathbf{R};\ x, y \in X). \tag{12}$$

Now if $\lambda = \sigma + i\tau$ ($\sigma, \tau \in \mathbf{R}$) then the biadditivity of L (Lemma 1) and (12) imply:

$$\begin{aligned} L(\lambda x, y) &= L(\sigma x + i\tau x, y) = L(\sigma x, y) + L(i\tau x, y) \\ &= \sigma L(x, y) + iL(\tau x, y) = \sigma L(x, y) + i\tau L(x, y) = \lambda\, L(x, y). \\ L(x, \lambda y) &= L(x, \sigma y + i\tau y) = L(x, \sigma y) + L(x, i\tau y) \\ &= L(x, \sigma y) - iL(x, \tau y) = L(\sigma x, y) - iL(\tau x, y) = \\ &= \sigma L(x, y) - i\tau L(x, y) = \overline{\lambda}\, L(x, y). \quad \square \end{aligned}$$

The complexification X_c of a real vector space X is defined as a set $X \times X$ with algebraic operations:

$$(x,y)+(x',y')=(x+x',y+y') \quad (x,x',y,y' \in X),$$
$$(\sigma+i\tau)(x,y)=(\sigma x-\tau y,\tau x+\sigma y') \quad (\sigma,\tau \in \mathbf{R};\ x,y \in X).$$

We write $(x,y)=x+iy \quad (x,y \in X)$.

If $B: X\times X \to R$ is a bilinear form then its complexification $B_c : X_c \times X_c \to C$ is defined by

$$B_c(x+iy,x'+iy')=B(x,x')+B(y,y')+i(B(y,x')-B(x,y')).$$

If B is symmetric then

$$B_c(x+iy,x+iy)=B(x,x)+B(y,y) \quad (x,y \in X)$$

THEOREM 2. *Let X be a real vector space and $q : X \to \mathbf{R}$ any function and $B : X \times X \to \mathbf{R}$ a function defined by*

$$B(x,y)=(q(x+y)-q(x-y))/4 \quad (x,y \in X) \tag{13}$$

Then, the function B is bilinear if and only if the compexification q_c:

$$q_c(x+iy)=q(x)+q(y) \quad (x,y \in X) \tag{14}$$

of q has the following homogeneity property

$$q_c(\lambda z)=|\lambda|^2 q_c(z) \quad (\lambda \in C;\ z \in X_c) \tag{15}$$

Proof. For $\lambda=\sigma+i\tau$ $(\sigma,\tau \in \mathbf{R})$ and $z=x+iy$ $(x,y \in X)$ from (15) we get:

$$(\sigma^2+\tau^2)(q(x)+q(y))=q(\sigma x-\tau y)+q(\tau x+\sigma y). \tag{16}$$

From (16) for $\sigma=\tau=1$ we find that q satisfies the parallelogram law (2) so that by Lemma 1 the function B is biadditive, symmetric and $q(x)=B(x,x)$. Furthermore (14) and (2) imply

$$q_c(u+v)+q_c(u-v)=2q_c(u)+2q_c(v)$$

for all $u,v \in X$. Now (17), (15) and Theorem 1 imply that the functional $L : X_c \times X_c \to C$ defined by

$$L(u,v)=(q_c(u+v)-q_c(u-v))/4+i(q_c(u+iv)-q_c(u-iv))/4$$

is sesquilinear on X_c. If $u=x$ and $v=y$ are vectors in X, then

$$L(x,y)=(q(x+y)+q(0)-q(x-y)-q(0))/4+i(q(x)+q(y)-q(x)-q(y))=B(x,y).$$

Thus $B(tx,y)=L(tx,y)=tL(x,y)$ holds for any $t \in \mathbf{R}$ and all $x,y \in X$. This and $B(x,y)=B(y,x)$ imply that B is bilinear. $\square$

Using Therem 1 we see that one can define a quadratic form on a complex vector space X as a function $q : X \to C$ which satisfies the parallelogram law (2) and has the homogeneity property (3).

According to Theorem 2 a quadratic form on a real vector space X can be defined as a function $q : X \to \mathbf{R}$ such that its complexification q_c defined by (14) satisfies the homogeneity property (15). As it is well-known for a function $q : X \to \mathbf{R}$ defined on a realvector space which satisfies the parallelogram law (2) and the homogeneity property

$$q(tx) = t^2 q(x) \quad (t \in \mathbf{R},\ x \in X)$$

in general there does not exist a bilinear form $B : X \times X \to R$ such that $q(x) = B(x,x)$ $(x \in X)$ (See: [2], [1]).

Remark 1. If $f, F : \mathbf{R} \to \mathbf{R}$ are additive functions such that

$$(18) \qquad f(t) = t^2 f(1/t), \qquad F(t) = -t^2 F(1/t), \qquad (t \in \mathbf{R},\ t \neq 0)$$

then $f(t) = f(1)t$ and $F(ts) = tF(s) + sF(t)$, i.e. f is continuous and F is a derivation on $\mathbf{R}$, hence F is not continuous unless $F = 0$.

On the other hand if $f, F : C \to C$ are additive and if

$$(19) \qquad f(\lambda) = |\lambda|^2 f(1/\lambda), \quad F(\lambda) = -|\lambda|^2 F(1/\lambda) \qquad (\lambda \in C,\ \lambda \neq 0)$$

then $f(\lambda) = f(1)\operatorname{Re}\lambda$ and $F(\lambda) = F(i)\operatorname{Im}\lambda$. In this case both functions f and F are continuous. In fact, if $F(\lambda) = -|\lambda|^2 F(1/\lambda)$ holds for all $\lambda \in C$, $\lambda \neq 0$, then a function $f_1(\lambda) = F(i\lambda)$ satisfies the condition

$$f_1(\lambda) = |\lambda|^2 f_1(1/\lambda) \qquad (\lambda \in C,\ \lambda \neq 0).$$

By (18) we are given essentially different conditions on functions f and F while conditions given by (19) can be transformed one to another.

2. Quadratic forms on modules over algebras

By X and X' we denote complex vector spaces and by A complex algebra with unit 1. We assume that the algebra A has the following regularity property (R):

For any $t \in A$ there exists a natural number n such that $t + n$ and $t + n + 1$ are invertible elements in A.

Furthermore we assume that X is a left modul over A and that X' is left and right modul over A.

THEOREM 3. *Let A, X and X' be as above. If $q : X \to X'$ is a quadratic form i.e.*

$$(1) \qquad q(x+y) + q(x-y) = 2q(x) + 2q(y) \quad (x, y \in X)$$

and if q satisfies the homogeneity condition

$$(2) \qquad q(tx) = tq(x)t \qquad (t \in A,\ x \in X)$$

then the function $M : X \times X \to X$ *defined by*

(3) $M(x, y) = (q(x + y) - q(x - y))/8 - i(q(x + iy) - q(x - iy))/8 \quad (x, y \in X)$

is biadditive, symmetric,

$$q(x) = M(x, x), \quad M(ix, y) = iM(x, y) \quad (x, y \in X)$$

and

(4) $$M(tx, y) + M(x, ty) = tM(x, y) + M(x, y)t \quad (t \in A;\ x, y \in X).$$

Furthermore, the function

(5) $$h(t; x, y) = (M(tx, y) - M(x, ty))/2 \quad (t \in A;\ x, y \in X)$$

is a Jordan derivation on A*, i.e.*

$$h(t \circ s; x, y) = t \circ h(s; x, y) + h(t; x, y) \circ s$$

holds true, where

$$t \circ s = ts + st \quad (t, s \in A).$$

The proof of Theorem 3 is obtained by using the following two lemmas.

LEMMA 4. (See Lemma 1 in [4]). *If an additive function* $g : A \to X'$ *for each invertible element* $t \in A$ *satisfies the condition*

(7) $$g(t) = tg(t^{-1})t$$

then

(8) $$g(t) = (tg(1) + g(1)t)/2 \quad (t \in A).$$

LEMMA 5. *If an additive function* $h : A \to X'$ *for each invertible element* $t \in A$ *satisfies the condition*

(7) $$h(t) = -th(t^{-1})t$$

then h *is a Jordan derivation on* A*, i.e.*

$$h(t \circ s) = h(t) \circ s + t \circ h(s) \quad (t, s \in A).$$

Proof of Lemma 5. For $t \in A$ we take a natural number n such that $t + n$ and $t + n + 1$ are invertible in A. By applying the function h on the identity

$$(t + n)^{-1} - (t + n + 1)^{-1} = (t^2 + 2nt + t + n^2 + n)^{-1}$$

and by using (9) we get

$$-(t + n)^{-1} \cdot h(t + n) \cdot (t + n)^{-1} + (t + n + 1)^{-1} \cdot h(t + n + 1) \cdot (t + n + 1)^{-1} =$$
$$= -(t^2 + 2nt + t + n^2 + n)^{-1} \cdot h(t^2 + 2nt + t + n^2 + n) \cdot (t^2 + 2nt + t + n^2 + n).$$

Mulltiply the last relation from the left and from the right by $(t+n)(t+n+1)$ to get:

$$h(t^2+2nt+tn^2+n)=(t+n+1)\cdot h(t+n)\cdot(t+n+1)-(t+n)\cdot h(t+n)\cdot(t+n)$$

from which by using $h(1)=0$ we get

$$h(t^2)=th(t)+h(t)t \quad (t\in A). \tag{11}$$

If in (11) we replace t by $t+s$ we get (10). □

Proof of Theorem 3. Since q is quadratic, the function M defined by (3) is biadditive and $q(x)=M(x,x)$. By using (2) it is easy to find

$$M(ix,y)=iM(x,y), \quad M(x,y)=M(y,x)$$

and

$$M(tx,y)=tM(x,t^{-1}y)t. \tag{12}$$

for any invertible element $t\in A$ and for all $x,y\in X$. If $x,y\in X$ are fixed, then the function

$$g(t)=M(tx,y)+M(x,ty) \quad (t\in A)$$

satisfies all conditions of Lemma 4 so that (8) and $g(1)=2M(x,y)$ imply (4).

By using (2) for the function h defined by (5) we find

$$h(t;x,y)-th(t^{-1};x,y)t$$

for any invertible $t\in A$. By applying Lemma 5 we get (6). □

Remark 2. If X and X' are real vector spaces and A is a real algebra with the regularity property (R), then for a quadratic form which has the homogeneity property (2) the function

$$B(x,y)=(q(x+y)-q(x-y))/4 \quad (x,y\in X)$$

is biadditive, symmetric, $B(x,x)=q(x) \quad (x\in X)$,

$$B(tx,y)+B(x,ty)=tB(x,y)+B(x,y) \quad (t\in A;\ x,y\in X)$$

and the function

$$h(t;x,y)=(B(tx,y)-B(x,ty))/4 \quad (t\in A;\ x,y\in X)$$

is a Jordan derivation, i.e. h has the property (6). The proof of these fact follows the proof of Theorem 3.

REFERENCES

[1] A. M. Gleason, *The definition of quadratic form*, Amer. Math. Monthly **73** (1966), 1049–1056.

[2] S. Kurepa, *The Cauchy functional equation and scalar product in vector spaces*, Glasnik Mat. Fiz. Astr. **19** (1964), 23–36.

[3] S. Kurepa, *Quadratic and sesquilinear functionals*, Glasnik Mat. Fiz. Astr. **20** (1965), 79–92.

[4] S. Kurepa, *On quadratic forms*, to appear in Aequationes Math.

[5] P. Vrbová, *Quadratic functionals and bilinear forms*, Časopis Pest. Mat. **98** (1973), 159–161.

[6] J. Vukan, *A result concerning additive functions in hermitian Banach $*$-algebras*, Proc. Amer. Math. Soc. **91** (1984), 367–372.

[7] J. Vukman, *Some results concerning the Cauchy functional equation in certain Banach algebras*, Bull. Austral. Math. Soc. **31** (l985), 137–144.

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