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ON GENERAL SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with the form of the general solution of the equation (1), and the Theorem of Section 1 gives an answer which is somewhat different from the classical Picard result. Not only is the proof elementary, but the requests for the coefficients are much less restrictive; see the assumption (A_0). On the other hand, we had to introduce the additional assumption (A_0). Several examples are constructed in order to throw more light on the importance of those two assumptions.

0. Consider the differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0, \tag{1}$$

where the real functions $a_1, \dots, a_n$ are defined on an open interval $I \subseteq R$. This supposition regarding the coefficients $a_1, \dots, a_n$ and the interval I will be assumed throughout this paper.

The well-known theorem (see, for example, [1], [2] or [3]) states that if all the functions $a_1, \dots, a_n$ are continuous on I, then there exists a (linearly independent) system $y_1, \dots, y_n$ of solutions of (1) on I such that the general solution of (1) is given by their linear combination, i.e. by

$$y(x) = \sum_{k-1}^{n} C_k y_k(x) \quad (x \in I;\ C_1, \dots, C_n \text{ arbitrary constants}).$$

The usual (standard) proof of this theorem is based upon the classical Picard's existence and uniqueness theorem; more precisely, upon its special case for linear differential equations.

This basic question regarding the form of the general solution of (1) is once again treated in this paper, and we give in the Theorem of Section 1 an answer which is somewhat different from the classical answer cited above. Namely, our requests for the coefficients $a_1, \dots, a_n$ are much less restrictive (assumption (A_1)),

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but we had to introduce the additional assumption (A_2). Besides, the proof is elementary. Sections 2 and 3 are devoted to the investigation of the effects of (A_1) and (A_2) on the Theorem. Several examples are constructed to show e.g. that the condition (A_1) cannot be omitted from the Theorem, and that the same conclusion is probably also true for (A_2). The facts established by those examples are then summarized in Propositions 1 and 2.

The Theorem of Section 1, together with its proof, may be used to provide a theoretical background for the representation (too often taken for granted) of general solutions of linear differential equations in elementary courses which avoid the existence and uniqueness theorems. Besides, when a system $y_1, \dots, y_n$ of solutions of (1) with the property (A_2) is found (by guesswork, inspection, or some other ad-hoc method, as it often happens in practice) according to our Theorem it may safely be concluded that (2) is the general solution of (1), provided that (A_1) is fulfilled.

1. As usual, we say that F is a primitive function of f on I, if $F'(x) = f(x)$ for all $x \in I$. If a function f has a primitive function on I, we shall say, as in [5], that it has the P_I-property. Similarly. following the usual practice, we say that a system $u_1, \dots, u_n$ of real functions defined on an open interval $I \subseteq R$ is linearly dependent if there exist constants $C_1, \dots, C_n$, not all zero, such that $\sum_{k=1}^n C_k u_k(x) = 0$ for all $x \in I$; otherwise, it is linearly independent. If we suppose that the functions $u_1, \dots, u_n$ have $(n-1)$-st derivatives on I then in order that they be linearly dependent it is necessary, but not sufficient, that their Wronskian vanishes for all $x \in I$, i.e. that

$$W(u_1, \dots, u_n) = \begin{vmatrix} u_1(x) & u_2(x) & u_n(x) \\ u_1'(x) & u_2'(x) & u_n'(x) \\ \vdots & & \\ u_1^{(n-1)}(x) & u_2^{(n-1)}(x) & u_n^{(n-1)}(x) \end{vmatrix} = 0 \quad (x \in I),$$

In the case when all the coefficients of the equation (1) are continuous on I, then by a known result (based on the mentioned existence and uniqueness theorem) the above condition is also sufficient for the system $u_1, \dots, u_n$ of n solutions of (1) to be linearly dependent.

In the proof of the main theorem and of the propositions which follow we shall use the following auxiliary results.

LEMMA 1. *For the equation*

$$y' + a(x)y = 0, \tag{3}$$

where the function a is defined on I, the following two conditions are equivalent:

(i) *the function a has the P_I-property.*

(ii) *the equation* (3) *has at least one solution y_0 such that $y_0(x) \neq 0$ for all $x \in I$. Each of the conditions* (i) *and* (ii) *implies that for every nontrivial solution y of the equation* (3) *we have $y(x) \neq 0$ for all $x \in I$.*

The following condition is effectively weaker than the above two:

(iii) *there exists a solution y_0 of* (3) *with the property that all solutions of* (3) *are given by*

$$y(x) = Cy_0(x) \qquad (x \in I;\ C \textit{ arbitrary constant}).$$

Proof. Suppose that A is a primitive function of a on I. Then for each function y differentiable on I, the function u defined by

$$y(x) = e^{-A(x)}u(x) \qquad (x \in I) \tag{4}$$

is differentiable on I. Substituting (4) into (3), we conclude that this function y, (3) is equivalent to $u'(x) = 0$ $(x \in I)$, which in turn is equivalent to $u(x) = C$ $(x \in I)$, where C is an arbitrary constant. Hence, y is a solution of (3) if and only if

$$y(x) = Ce^{-A(x)} \qquad (x \in I;\ C = \text{const}), \tag{5}$$

which can be written in the form $y(x) = Cy_0(x)$ $(x \in I;\ C$ arbitrary constant), where $y_0(x) = e^{-A(x)} \neq 0$ $(x \in I)$. This proves the implication (i) $\Rightarrow$ (ii) and also (i) $\Rightarrow$ (iii).

If the equation (3) has a solution y_0 such that $y_0(x) \neq 0$ $(x \in I)$, then

$$a(x) = -y_0'(x)/y_0(x) = -(\log|y_0(x)|)' \qquad (x \in I),$$

(ii) $\Rightarrow$ (i).

The next assertion of Lemma 1 follows directly from (5).

Finally, in order to shaw that (iii) is weaker than (i), or (ii), let $I = \mathbf{R}$, and define the function a on $\mathbf{R}$ by:

$$a(x) = -1/x^2 \quad (x > 0), \qquad a(x) = 0 \quad (x \leq 0) \tag{6}$$

In this case the equation (3) has the solution y_0 defined by

$$y_0(x) = \exp(-1/x) \quad (x > 0), \qquad y_0(x) = 0 \quad (x \leq 0),$$

which is easily verified. If y is any solution of (3), then

$$y(x) = C\exp(-1/x) \quad (x > 0), \qquad y(x) = D \quad (x \leq 0),$$

where C, D are constant. But $\lim\limits_{x \to 0+} y(x) = 0 = D$, and hence $y(x) = Cy_0(x)$ $(x \in I)$, where C is an arbitrary constant. The condition (iii) is therefore fulfilled, but the function a defined by (6) does not have P_R-property (by Darboux's theorem, or because y_0 vanishes for some $x \in \mathbf{R}$, but is not identically zero).

LEMMA 2. (i) *For any system*

$$y_1, \ldots, y_n \tag{7}$$

of solutions of (1), *we have*

$$\frac{d}{dx}W(y_1, \ldots, y_n) + a_1(x)W(y_1, \ldots, y_n) = 0 \qquad (x \in I).$$

(ii) *If there exists a system* (7) *of solutions of* (1), *such that*

$$W(y_1, \ldots, y_n) \neq 0 \quad \textit{for all} \quad x \in I, \tag{8}$$

then the function a_1 has the P_I-property. Conversely, if a_1 has the P_I-property and if there exists a system (7) *such that $W(y_1, \ldots, y_n) \neq 0$ for at least one $x \in I$, then this system of solutions satisfies* (8).

(iii) *If a_1 has the P_I-property, then the Wronskian of an arbitrary system* (7) *of solutions* (1) is given by

$$W(y_1, \ldots, y_n) = CW(y_1^0, \ldots, y_n^0) \qquad (x \in I)$$

where $y_1^0, \ldots, y_n^0$ is one particular system of solutions and C is a constant.

Proof. Statement (i) is proved in the usual manner. Statements (ii) and (iii) follow from (i) and Lemma 1.

We now formulate and prove the main result of this paper.

THEOREM. *Suppose that:*
(A_1) *the function a_1 has the P_I-property;*
(A_1) *there exists a system* (7) *of solutions of* (1) *on I such that*

$$W(y_1, \ldots, y_n) \neq 0$$

for at least one $x \in I$.

Then the general solution of the equation (1) *is given by*

$$y(x) = \sum_{k=1}^{n} C_k y_k(x) \quad (x \in I;\ C_1, \ldots C_n \textit{ arbitrary constants}).$$

Proof. Suppose that the conditions (A_1) and (A_2) are satisfied, and that y is a solution of the equation (1) on I. We have

$$0 = \begin{vmatrix} y(x) & y_1(x) & \ldots & y_n(x) \\ y(x) & y_1(x) & & y_n(x) \\ y'(x) & y_1'(x) & & y_n'(x) \\ \vdots & & & \\ y^{(n-1)}(x) & y_1^{(n-1)}(x) & & y_n^{(n-1)}(x) \end{vmatrix} =$$

$$= y(x)W(y_1, \ldots, y_n) + \sum_{k=1}^{n} (-1)^k y_k(x) W(y, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n) \quad (x \in I).$$

Furthermore, in virtue of Lemma 2, (8) is valid and we obtain

$$y(x) = \sum_{k=1}^{n} (-1)^k \frac{W(y, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)}{W(y_1, \ldots, y_n)} y_k(x) \quad (x \in I).$$

Again, according to Lemma 2, the coefficients of $y_1, \ldots, y_n$ are constants, and therefore

$$y(x) = \sum_{k=1}^{n} C_k y_k(x) \quad (x \in I;\ C_1, \ldots C_n = \text{const.}).$$

Conversely, it is readily verified that any function y, defined by (9) is a solution of the equation (1) on I.

Remark 1. The above theorem and its proof were given for $n = 2$, in a rudimentary form, in [4].

2. Now that the Theorem is established, we see that in the proof given above we made use of the assumptions (A_1) and (A_2). The question for each of those assumptions is, of course, whether it is essential for the validity of the theorem, and also what happens if it is suppressed. The Proposition 1 given at the end of this section provides an answer for the assumptions (A_1). It is based upon the following four examples.

Example 1. Consider the equation

$$y^{(n)} + a(x)y^{(n-1)}(x) = 0 \quad (x \in \mathbf{R}) \tag{10}$$

where the function a is defined on $\mathbf{R}$ by (6). As we know (proof of Lemma 1) this function does not have the P_R-property. On the other hand, the equation (10) has solutions $y_1, \dots, y_n$ on $\mathbf{R}$ defined by

$$\begin{aligned} &y_1(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} \alpha(t)dt \\ &y_2(x) = 1,\ y_3(x) = x,\ \dots\ ,\ y_n(x) = x^{n-2} \end{aligned} \qquad (x \in \mathbf{R})$$

where $\alpha(x) = e^{-1/x}$ $(x > 0)$, $\alpha(x) = 0$ $(x \le 0)$, which are linearly independent on $\mathbf{R}$. Indeed, the functions $y_2, \dots, y_n$ are clearly independent on $\mathbf{R}$, whereas linear dependence of the form

$$C_1 y_1(x) + \sum_{k=2}^{n} C_k x^{k-2} = \quad (x \in \mathbf{R};\ C_1 \ne 0)$$

would imply, after $n-1$ differentiations, that $y_1^{(n-1)}(x)e^{-1/x} = 0$ $(x > 0)$, which is absurd.

The Wronskian of this system vanishes for $x = 0$, since $y_1(0) = y_1'(0) = \cdots = y_1^{(n-1)}(0) = 0$, but its value differs from 0 for $x > 0$, since all the functions form a linearly independent system on $(0, +\infty)$, and the coefficients of (10) are continous for $x > 0$. Clearly, all the solutions of the equation (10) are given by

$$y(x) = \sum_{k=2}^{n} C_k y_k(x) \quad (x \in \mathbf{R};\ C_1, \dots, C_n = \text{const.}).$$

Example 2. We now consider the equation

$$y^{(n)} + b(x)y^{(n-1)} = 0 \quad (x \in \mathbf{R}) \tag{11}$$

where b is defined on $\mathbf{R}$ by

$$b(x) = -x^{-2}\operatorname{sgn} x\ (x \ne 0), \quad b(0) = 0, \tag{12}$$

and again does not have the P_R-property. The functions $y_1, \dots y_n$ defined on $\mathbf{R}$ by

$$
\begin{aligned}
&y_1(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} \beta(t)dt \\
&y_2(x) = 1, y_3(x) = x, \dots, y_n(x) = x^{n-2} \quad (x \in \mathbf{R})
\end{aligned} \tag{13}
$$

with $\beta(x) = \exp(x^{-1}\operatorname{sgn} x)$ $(x \neq 0)$, $\beta(0) = 0$, are linearly independent solutions of (11). Their Wronskian vanishes at $x = 0$, and does not vanish for $x > 0$, but in this case the general solutions of (11) cannot be written in the form (9), since there is no choise of constants $C_1, \dots, C_n$ which leads to the following solution:

$$
y_0(x) = (\operatorname{sgn} x) y_1(x) \tag{14}
$$

of (11).

Example 3. We now extend the above example to show tha it is not possible to construct a linearly independent system $Y_1, \dots, Y_n$ of solutions of (11) such that its general solution is $y(x) = \sum_{k=1}^n C_k Y_k(x)$ with constants $C_1, \dots, C_n$. Indeed, since the general solution of (11) is given by

$$
y(x) = \begin{cases} C_1^{(1)} y_1(x) + \displaystyle\sum_{k=2}^n C_k y_k(x) & (x \geq 0) \\ C_1^{(2)} y_1(x) + \displaystyle\sum_{k=2}^n C_k y_k(x) & (x < 0) \end{cases} \tag{15}
$$

where $C_1^{(1)}, C_1^{(2)}, C_2, \dots, C_n$ are arbitrary constants, anyone of solutions Y_ν of that equation must have the form

$$
Y_\nu(x) = \begin{cases} C_{1,\nu}^{(1)} y_1(x) + \displaystyle\sum_{k=2}^n C_{k,\nu} y_k(x) & (x \geq 0) \\ C_{1,\nu}^{(2)} y_1(x) + \displaystyle\sum_{k=2}^n C_{k,\nu} y_k(x) & (x < 0) \end{cases} \qquad (\nu = 1, \dots, n)
$$

for some constants $C_{1,\nu}^{(1)}, C_{1,\nu}^{(2)}, C_{k,\nu}$ $(k = 2, \dots, n;\ \nu = 1, \dots, n)$. If a linear combination of those functions, namely if

$$
Y(x) = \sum_{\nu=1}^n C_\nu Y_\nu(x) = \begin{cases} y_1(x) \sum_{\nu=1}^n C_\nu C_{1,\nu}^{(1)} + \displaystyle\sum_{k=2}^n y_k(x) \sum_{\nu=1}^n C_\nu C_{k,\nu} & (x \geq 0) \\ y_1(x) \sum_{\nu=1}^n C_\nu C_{1,\nu}^{(2)} + \displaystyle\sum_{k=2}^n y_k(x) \sum_{\nu=1}^n C_\nu C_{k,\nu} & (x < 0), \end{cases}
$$

where the general solution of (11), then for any real numbers $D_1^{(1)}, D_1^{(2)}, D_2, \dots, D_n$ there would have to exist such values of the constants $C_1, \dots, C_n$ to ensure that

$$
Y(x) = \begin{cases} D_1^{(1)} y_1(x) + \displaystyle\sum_{k=2}^n D_k y_k(x) & (x > 0); \\ D_1^{(2)} y_1(x) + \displaystyle\sum_{k=2}^n D_k y_k(x) & (x < 0). \end{cases}
$$

Since the functions $y_1, \ldots, y_n$ are linearly independent both on $(0, +\infty)$ and $(-\infty, 0)$, this means that there would have to exist $C_1, \ldots, C_n$ such that

$$\sum_{\nu=1}^{n} C_\nu C_{1,\nu}^{(1)} = D_1^{(1)}, \ \sum_{\nu=1}^{n} C_\nu C_{1,\nu}^{(2)} = D_1^{(2)}, \ \sum_{\nu=1}^{n} C_\nu C_{k,\nu} = D_k \quad (k = 2, \ldots, n).$$

However this system of $n+1$ equations in n unknowns $C_1, \ldots, C_n$ need not have solutions for arbitrary given numbers $D_1^{(1)}, D_1^{(2)}, D_2, \ldots, D_n$, no matter how the numbers $C_{1,\nu}^{(1)}, C_{1,\nu}^{(2)}, C_{k,\nu}$ $(k = 2, \ldots, n;\ \nu = 1, \ldots, n)$ are previously chosen, implying that the general solution of (11) cannot be written in the form (9).

This example also shows that *the assumption* (A_1) *cannot be omitted from the following slightly weaker form of our Theorem:* (T) *Under the conditions* (A_1) *and* (A_2) *there exists a system of* n *solutions of* (1) *such that the general solution of* (1) *is the linear combination of functions of this system.*

Example 4. We again use the equation (11), with (12), and the solutions $y_0, y_1, \ldots, y_{n-1}$ defined by (13) and (14) to establish one more possibility. Namely, since

$$\begin{aligned}(-1)y_0(x) + 1 \cdot y_1(x) + 0 \cdot y_2(x) + \cdots + 0 \cdot y_{n-1}(x) = 0 \quad (x > 0),\\ 1 \cdot y_0(x) + 1 \cdot y_1(x) + 0 \cdot y_2(x) + \cdots + 0 \cdot y_{n-1}(x) = 0 \quad (x < 0)\end{aligned}$$

the Wronskian of this system vanishes for $x \neq 0$. But we also have $y_1(0) = y_1'(0) = \cdots = y_1^{(n-1)}(0) = 0$, and so it also vanishes for $x = 0$. In order to show that this system is linearly independent, suppose that there exist constants $C_1, \ldots, C_n$, not all zero, such that

$$C_1 y_0(x) + C_2 y_1(x) + \cdots + C_n y_{n-1}(x) = 0 \quad (x > 0), (x \in R). \tag{16}$$

Since $y_0, y_1, \ldots, y_{n-1}$ are linearly independent, this would imply that

$$C_1^2 + C_2^2 > 0. \tag{17}$$

On the other hand, (16) can be split into

$$\begin{aligned}(C_1 + C_2)y_1(x) + C_3 y_2(x) + \cdots + C_n y_{n-1}(x) \quad (x > 0),\\ (-C_1 + C_2)y_1(x) + C_3 y_2(x) + \cdots + C_n y_{n-1}(x) \quad (x < 0)\end{aligned}$$

and differentiating the last two equalities $n-1$ times we get

$$(C_1 + C_2)e^{-1/x} = 0 \quad (x > 0); \quad (-C_1 + C_2)e^{-1/x} = 0 \quad (x < 0)$$

and consequently $C_1 + C_= - C_1 + C_2 = 0$, i.e. $C_1 = C_2 = 0$, contradicting (17).

The conclusions established by the preceeding examples are now combined into the following.

Proposition 1. *Suppose that* $n \in N$ *and that the function* a_1 *does not have the* P_I*-property.*

(i) *The equation* (1) *may* (Example 1) *or may not* (Example 3) *have a system of solutions* (7) *such that its general solution is given by* (9).

(ii) *There may exist a system of solutions* (7) *of the equation* (1) *whose Wronskian vanishes at some, and does not vanish at other points of* I. *Besides, in this case each of the following two cases is possible:*

(a) *the general solution of* (1) *is given by* (9) (Example 1);

(b) *the general solution cannot be written in the form* (9) (Example 2).

(iii) *There may exist a linearly independent system of* n *solutions of the equation* (1) *whose Wronskian vanishes for all* $x \in X$ (Example 4)

3. In the preceeding section we have examined various situations which may arise if the assertion (A_1) is dropped. Conversely, we now assume that (A_1) is fulfilled, and we give two examples to illustrate two possibilities which may take place in this case.

Example 5. Consider the equation

$$y^{(n)} + d(x)y = 0 \qquad (n \geq 2;\ x \in R), \tag{18}$$

where d is the Dirichlet function, i.e. $d(x) = 1$ (x rational) and $d(x) = 0$ (x irrational). The coefficient a_1 of $y^{(n-1)}$ in (18) has the P_R-property. We prove that the unique solution of (18) is given by $y_0(x) = 0$ $(x \in R)$.

Conversely, suppose that there exists a solution Y of (18) such that $Y(x_0) > 0$, say, for some $x_0 \in R$. But then there would exist an interval $J = (\alpha, \beta) \ni x_0$ such that $Y(x) > Y(x_0)/2$ $(x \in J)$ and hence the set of values which the function $d(x)Y(x)$ takes when $x \in J$ would be equal to $S \cup \{0\}$, where $\emptyset \neq S \subset (Y(x_0)/2, \infty)$. This would mean that the function $d(x)Y(x) = -(Y^{(n-1)}(x))'$ does not have the P_R-property, which is absurd. Hence, the unique solution of (18) is $y_0(x) = 0$ $(x \in R)$, and so any system of solutions of that equation must be linearly dependent.

Example 6. If $c(x) = -6/x^2$ $(x \neq 0)$, $c(0) = 0$, then for $n \geq 2$, the coefficient of $y^{(n-1)}$ in the equation

$$y^{(n)} + c(x)y^{(n-2)} = 0 \quad (x \in R) \tag{19}$$

has the P_R-property. This equation has the system of solutions

$$y_1(x) = x^{n+1},\ y_2(x) = (\operatorname{sgn} x)\, x^{n+1},\ y_3(x) = 1,\ y_n(x) = x^{n-3} \quad (x \in R) \tag{20}$$

which is linearly independent on R, but whose Wronskian vanishes for all $x \in R$. Indeed, since

$$\begin{gathered} 1 \cdot y_1(x) + (-1)y_2(x) + 0 \cdot y_3(x) + \cdots + 0 \cdot y_n(x) = 0 \quad (x > 0) \\ 1 \cdot y_1(x) + 1 \cdot y_2(x) + 0 \cdot y_3(x) + \cdots + 0 \cdot y_n(x) = 0 \quad (x < 0) \end{gathered}$$

we conclude that $W(y_1, \ldots, y_n) = 0$ for all $x \neq 0$. On the other hand, $y_1(0) = y_1'(0) = \cdots = y_1^{(n-1)}(0) = 0$, implying that $W(y_1, \ldots, y_n) = 0$ for $x = 0$. In order to show that this system is linearly independent, put $\sum_{k=1}^{n} C_k y_k(x) = 0$ $(x \in R)$ where $C_1, \dots, C_n$ are constants. This splits into

$$(C_1 + C_2)x^{n+1} + C_3 + \cdots + C_n x^{n-3} = 0 \quad (x > 0)$$
$$(C_1 - C_2)x^{n+1} + C_3 + \cdots + C_n x^{n-3} = 0 \quad (x < 0)$$

and so $C_1 + C_2 = C_1 - C_2 = C_3 = \cdots = C_n = 0$, i.e. $C_1 = C_2 = \cdots = C_n = 0$.

We now construct the general solution of (19) using the system (20). Consider first the case when $n = 2$, i.e. the equation

$$y'' + c(x)y = 0 \quad (x \in R). \tag{21}$$

For $x > 0$ this equation has linearly independent solutions y_1, y_2 defined by $y_1(x) = x^3$, $y_2(x) = 1/x^2$, and so by the standard result (or by our Theorem) all the solutions on $(0, +\infty)$ of this equation are given by

$$y(x) = K_1 x^3 + K_2/x^2 \qquad (x > 0), \tag{22}$$

where K_1, K_2 are arbitrary constants. This means that the restrictions on $(0, +\infty)$ of all the solutions of the equation (21) are among the functions (22). But those solutions must be continuous at $x = 0$, and so $K_2 = 0$. Hence, any solution of (22) for $x > 0$, and also for ≥ 0, must be given by $y(x) = K^{(1)}x^3$ $(x \geq 0)$, where $K^{(1)}$ is a constant. Similarly, the restrictions on $(-\infty, 0)$ of all the solutions of (21) must be given by $y(x) = K^{(2)}x^3$ $(x < 0)$, where $K^{(2)}$ is a constant. Therefore, the general solution of (21) is given by

$$y(x) = \begin{cases} K^{(1)}x^3 & (x \geq 0) \\ K^{(2)}x^3 & (x < 0) \end{cases} (K^{(1)}, K^{(2)} = \text{const}).$$

Now for any $n \geq 2$, if y is a solution of (19), then

$$y^{(n-2)}(x) = z(x)\ (x \in R) \quad \wedge \quad z'' + c(x) = 0\ (x \in R),$$

i.e.

$$y^{(n-2)}(x) = z(x)\ (x \in R) \quad \wedge \quad z(x) = \begin{cases} K^{(1)}x^3 & (x \geq 0) \\ K^{(2)}x^3 & (x < 0) \end{cases}$$

wherefrom we obtain the general solution of (l9):

$$y(x) = \begin{cases} C^{(1)}x^{n+1} + C_3 + C_4 x + \cdot + C_n x^{n-3} & (x \geq 0) \\ C^{(2)}x^{n+1} + C_3 + C_4 x + \cdot + C_n x^{n-3} & (x < 0) \end{cases}$$

where $C^{(1)}, C^{(2)}, C_3, \dots, C_n$ are arbitrary constants. However, it is clear that this solution can also be written in the form $y(x) = \sum_{k=1}^{n} C_k y_k(x)$ $(x \in R)$, where $C_1, \dots, C_n$ are arbitrary constants and $y_1, \dots, y_n$ are defined by (20).

As before, we combine the conclusion of the last two examples into the following

Proposition 2. *Suppose that al has the P_I-property. For any $n \geq 2$ it is possible that:*

(i) *any system of n solutions of* (1) *is linearly dependent;*

(ii) *there exists a system of n linearly independent solutions of* (1) *whose Wronskian vanishes for all $x \in I$, but the general solution of* (1) *is again given by* (9).

The above proposition implies that (A_2) is not a consequence of (A_1).

It would be interesting to investigate whether it is possible that the function a_1 has the P_I-property, and that at the same time there is no system $y_1, \dots, y_n$ of solutions of (1) such that the general solution of (1) is given by (9); in other words whether (A_2) may, or may not be omitted from the statement of the Theorem; strictly speaking, from the weaker form of this theorem, denoted earlier by (T). *This question remains open.*

Remark 2. The following example shows that *for any $n \geq 2$, (A_2) can be realized when the condition (A_1) is satisfied and the condition of continuity of all the coeficients* (i.e. the condition of the classical theorem cited at the beginning) *is not.*

Let a_1 have P_I-propqrty, not being continous on I. Then the equation $y^{(n)} + a_1(x)y^{(n-1)} = 0$, for any $n \geq 2$, has the following system of n solutions

$$\int_{x_0}^{x} dx_1 \int_{x_0}^{x_2} dx_2 \cdots \int_{x_0}^{x_{n-2}} e^{-A(t)} dt,\ 1,\ x,\ \dots\ ,\ x^{n-2},$$

whose Wronskian differs from 0 on I. Here A is a primitive function of a_1 on I and x_0 is a point of I.

4. Needless to say, analogous results hold for the nonhomogeneous equations of the form

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x)$$

where the function F is defined on I. The Theorem is also carried over to first order linear systems of differential equations.

REFERENCES

[1] E. L. Ince, *Ordinary differential equations*, New York, 1945.

[2] Х. Г. Петровский, *Лекции по теории обыкновенных дифференциальных уравнений*, Наука, Москва, 1970.

[3] E. Kamke, *Differentialgleichungen — Lösungsmethoden und Lösungen*, I. 6. verbesserte Auflage, Leipzig, 1959.

[4] J. D. Kečkić, *Reproductivity of some equation of analysis*, Publ. Inst. Math. (Beograd) **31** (**45**) (1982), 73–81.

[5] D. D. Adamović, *Quelques remarques relatives aux fonctions primitives des fonctions réelles*, to appear.

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