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EXTENSIONS OF SOME FIXED POINT THEOREMS OF RHOADES, ĆIRIĆ, MAITI AND PAL

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In a recent paper Rhoades [6] has shown, for a selfmap T of a Banach space satisfying the contractive definitions of Ćirić [1] or of Pal and Maiti [5], that if the sequence of Mann iterates converges then it converges to a fixed point of T. In this note we propose to draw the same conclusion in some of these cases even for subsequential limit points, i. e. every subsequential limit point of the sequence of Mann iterates will be a fixed point of T. Further we shall derive the conclusions of Rhoades in the case of mappings satisfying even weaker conditions. Our final result will be concerned with the extension of a result of Maiti and Babu [4] to mappings satisfying conditions similar to those in Rhoades [6, Theorem 3]. This is closed in spirit to the main result of Diaz and Metcalf [2].

Let T be a selfmapping of a Banach space X. The Mann iterative process associated with T is defined in the following way. Let $x_0 \in X$, and set $x_{n+1} = (1-c_n)x_n + c_n T x_n$ for $n > 0$, where $\{c_n\}$ satisfies: (i) $c_0 = 1$, (ii) $0 < c_n \leq 1$ for $n > 0$, (iii) $\sum c_n$ diverges. In this note we place the additional restriction that (iv) $\lim\limits_{n\to\infty} c_n = h > 0$.

A generalization of a contractive definition of Ćirić which has been used by Rhoades [6] is

(1) $d(Tx,Ty) \leq q \max\{cd(x,y), d(x,Tx) + d(y,Ty), d(x,Ty) + (y,Tx)\}$ where $c > 0$, $0 \leq q < 1$.

We shall use, instead, a mapping T satisfying

(2) $d(Tx,Ty) \leq q \max\{cd(x,y) + d(x,Tx) + d(y,Ty), cd(x,y) + d(x,Ty) + {}$ ${}+d(y,Tx)\}$

and show that the results of Rhoades carry over to such mappings.

It is clear that (1) $\Rightarrow$ (2). To see that the reverse implication may not be true, consider $X = \{x,y,z\}$, $Tx = y$, $Ty = z$, $Tz = z$, $d(x,y) = 1.7$, $d(x,z) = 1.8$,

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$d(y,z) = 1.3$, $q = 0.28$, $c = 1$. Clearly (X,d) is a metric space and T is a selfmap satisfying (2), but not (1), because $1.3 = d(y,z) = d(Tx,Ty)$ and

$$q \max\{d(x,y), d(x,Tx) + d(Y,Ty), d(x,Ty) + d(y,Tx)\} = \\ = .28 \max\{1.7, 1.7 + 1.3, 1, 8 + 0\} = 0.84.$$

We now prove Rhoades' theorem in the case of mappings satisfying (2). This we state as our

THEOREM 1. *Let X be a closed convex subset of a normed space, T a selfmapping of X satisfying* (2) *on X, $\{x_n\}$ the sequence of Mann iterates associated with T, where $\{c_n\}$ satisfies* (*i*), (*ii*) *and* (*iv*). *If $\{x_n\}$ converges in X, then it converges to a fixed point of T.*

Proof. Let $z \in X$ satisfy $\lim_{n\to\infty} x_n = z$. Then

$$d(z,Tz) \le d(z,x_{n+1}) + d(x_{n+1},Tz) \le d(z,x_{n+1}) + (1-c_n)d(x_n,Tz) + c_n d(Tx_n,Tz) \le \\ \le d(z,x_{n+1}) + (1-c_n)d(x_n,Tz) + c_n q \cdot \max\{cd(x_n,z) + d(x_n,Tx_n) + \\ + d(z,Tz), cd(x_n,z) + d(x_n,Tz) + d(z,Tx_n)\}.$$

Using $d(x_n,x_{n+1}) = c_n d(x_n,Tx_n)$ and

$$d(z,Tx_n) \le d(z,x_n) + d(x_n,Tx_n) = d(z,X_n) + d(x_n,x_{n1})/c_n$$

and letting $n \to \infty$ we get

$$d(z,Tz) \le (1-h)d(z,Tz) + hg \max\{d(z,Tz), d(z,Tz)\} = (1-h+hq)d(z,Tz)$$

which is absurd since $q < 1$ unless $d(z,Tz) = 0$. Thus $z = Tz$ and z is a fixed point of T.

Pal and Maiti [5] have studied mappings T satisfying (3) For each $x,y \in X$, at least one of the following conditions holds:

(a) $d(x,Tx) + d(y,Ty) \le \alpha d(x,y)$, $1 \le \alpha < 2$,
(b) $d(x,Tx) + d(y,Ty) \le \beta\{d(x,Ty) + d(y,Tx) + d(x,y)\}$, $1/2 \le \beta < 2/3$,
(c) $d(x,Tx) + d(y,Ty) + d(Tx,Ty) \le \gamma\{d(x,Ty) + d(y,Tx)\}$, $1 \le \gamma < 3/2$,
(d) $d(Tx,Ty) \le \delta \max\{d(x,y), d(x,Tx), d(y,Ty), (d(x,Ty) + d(y,Tx))/2\}$, $0 < \delta < 1$.

We now improve Theorem 2 of Rhoades [6] by showing that the subsequential limit points of the sequence of Mann iterates are also fixed points of T.

THEOREM 2. *Let X be a Banach space, T a selfmapping of X satisfying* (3). *Let $\{x_n\}$ be the sequence of Mann iterates associated with $\{c_n\}$ satisfying* (*i*), (*ii*), *and* (*iv*). *Then the subsequential limit points of $\{x_n\}$ are fixed points of T. We assume that in case T satisfies* (3) (*c*), $\gamma > 1 + h/2$ *and $\delta < h$ in case T satisfies* (3) (*d*).

Proof. Let $x_{n_i} \to \xi$ as $i \to \infty$. In case T satisfies (3)(a) putting $x = x_n$ and $y = x_{n+1}$ we get

$$d(x_n, Tx_n) + d(x_{n+1} + Tx_{n+1}) \leq c_n d(x_n, Tx_n), \text{ or } d(x_{n+1}, Tx_{n+1}) \leq (\alpha c_n - 1) d(x_n, Tx_n).$$

Now $1 \leq \alpha < 2$ and $0 < c_n \leq 1$ implies $-1 < \alpha c_n - 1 < 1$. If for some n, $\alpha c_n - 1 \leq 0$ we shall have $d(x_{n+1}, Tx_{n+1}) = 0$ or $x_{n+1} = Tx_{n+1}$. Now $x_{n+2} = (1 - c_{n+1})x_{n+1} + c_{n+1}Tx_{n+1} = x_{n+1}$. Proceeding similarly $x_{n+1} = x_{n+i}$ for all $i \geq 1$ and $\xi = x_{n+1}$ and $\xi = T\xi$. Hence we shall assume that for all n, $\alpha c_n - 1 > 0$. Now $\alpha c_n - 1 \to \alpha h - 1$ as $n \to \infty$. Putting $\lambda_1 = \alpha h - 1$ we see that $0 \leq \lambda_1 < 1$ since $\alpha < 2$ and $h \leq 1$, and so we can find an integer n_{01} such that for all $n \geq n_{01}$, $\alpha c_n - 1 > (1 + \lambda_1)/2$. In case T satisfies (3)(b), putting $x = x_n$, $y = x_{n+1}$ we get

$$\begin{aligned} d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) &\leq \beta\{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n) + d(x_n, x_{n+1})\} \leq \\ &\leq \beta\{d(x_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1}) + (1 - c_n)d(x_n, Tx_n) + c_n d(x_n, Tx_n)\} = \\ &= \beta\{c_n d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) + d(x_n, Tx_n)\} \end{aligned}$$

or,

$$d(x_{n+1}, Tx_{n+1}) \leq \frac{\beta(1 + c_n) - 1}{1 - \beta} d(x_n, Tx_n).$$

Now,

$$\lim_{n \to \infty} (\beta(1 + c_n) - 1)/(1 - \beta) = (\beta(1 + h) - 1)/(1 - \beta)$$

and is < 1 iff $\beta(1 + h) - 1 < 1 - \beta$ or iff $\beta < 2/(2 + h)$ and, since $0 < h \leq 1$, we have $2/3 \leq 2/(2 + h) < 1$. Since $\beta < 2/3$ by hypothesis, taking $\lambda_2 = \beta(1 + h) - 1/(1 - \beta)$, we can find n_{02} such that for $n \geq n_{02}$, $(\beta(1 + c_n) - 1)/(1 - \beta) < (1 + \lambda_2)/2$. Now consider the case when the mapping T satisfies (3) (c). Putting $x = x_n$, $y = x_{n+1}$, we get

$$\begin{aligned} d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) + d(Tx_n, Tx_{n+1}) &\leq \gamma\{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)\} \leq \\ &\leq \gamma\{d(x_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1}) + (1 - c_n)d(x_n, Tx_n)\} = \\ &= \gamma\{c_n d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) + (1 - c_n)d(x_n, Tx_n)\} = \\ &= \gamma\{c_n d(x_{n+1}, Tx_{n+1}) + d(x_n, Tx_n)\}. \end{aligned}$$

Thus $d(Tx_n, Tx_{n+1}) \leq (\gamma - 1)\{d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\}$. Hence,

$$\begin{gathered} d(x_{n+1}, Tx_{n+1}) - d(x_{n+1}, Tx_n) \leq (\gamma - 1)\{d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\} \text{ or,} \\ d(x_{n+1}, Tx_{n+1}) - (1 - c_n)d(x_n, Tx_n) \leq (\gamma - 1)\{d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\} \text{ or,} \\ (2 - \gamma)d(x_{n+1}, Tx_{n+1}) \leq (\gamma - 1 + 1 - c_n)d(x_n, Tx_n) = (\gamma - c_n)d(x_n, Tx_n) \text{ or,} \\ d(x_{n+1}, Tx_{n+1}) \leq (\gamma - c_n)/(2 - \gamma) \cdot d(x_n, Tx_n). \end{gathered}$$

Now, $(\gamma - c_n)/(2 - \gamma) \to (\gamma - h)/(2 - \gamma)$ and $\dfrac{\gamma - h}{2 - \gamma} < 1$ if $\gamma - h < 2 - \gamma$ or iff $2\gamma < 2 + h$ or, iff $\gamma < 1 + h/2$, which is true. Therefore denoting $(\gamma - h)/(2 + \gamma)$ by $\lambda_3 < 1$, we can find n_{03} such that for $n \geq n_{03}$, $(\gamma - c_n)/(2 - \gamma) < (1 + \lambda_3)/2$. In case T satisfies (3) (d), putting $x = x_n$, $y = x_{n+1}$ we get

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) \leq \delta \max\{&d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ &1/2 \cdot [d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx)]\}, \end{aligned}$$

Since $d(x_n, x_{n+1}) = c_n d(x_n, Tx_n)$ and

$$d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1}) + (1 - c_n)d(x_n, Tx_n) =$$
$$= c_n d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) + (1 - c_n)d(x_n, Tx_n) =$$
$$= d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})$$

we have

$$d(Tx_n, Tx_{n+1}) \leq \delta \max\{c_n d(x_n, Tx_n), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), 1/2[d(x_n, Tx_n)+$$
$$+ d(x_{n+1}, Tx_{n+1})]\} = \delta \max\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), 1/2[d(x_n, Tx_n)+$$
$$+ d(x_{n+1}, Tx_{n+1})]\} = \delta \max\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\}.$$

Now

$$d(Tx_n, Tx_{n+1}) \geq d(x_{n+1}, Tx_{n+1}) - d(x_{n+1}, Tx_n) = d(x_{n+1}, Tx_{n+1}) - (1 - c_n)d(x_n, Tx_n).$$

In case

$$d(x_{n+1}, Tx_{n+1}) - (1 - c_n)d(x_n, Tx_n) \leq \delta d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}) \leq$$
$$\leq (1 + \delta - c_n)d(x_n, Tx_n),$$

and in case $d(x_{n+1}, Tx_{n+1}) - (1 - c_n)d(x_n, Tx_n) \leq \delta d(x_{n+1}, Tx_{n+1})$, we get

$$(1 - \delta)d(x_{n+1}, Tx_{n1}) \leq (1 - c_n)d(x_n, Tx_n) \quad \text{or}$$
$$d(x_{n+1}, Tx_{n+1}) \leq (1 - c_n)/(1 - \delta) \cdot d(x_n, Tx_n).$$

Therefore, $d(x_{n+1}, Tx_{n+1}) \leq \max(1 + \delta + c_n, (1 + c_n)/(1 - \delta) \cdot d(x_n, Tx_n))$. Now as $n \to \infty$,

$$\max(1 + \delta - c_n, (1 - c_n)/(1 - \delta)) \to \max(1 + \delta - h, (1 - h)/(1 - \delta))$$

and $1 + \delta - h < 1$ iff $h > \delta$ and $(1 - h)/(1 - \delta) < 1$ iff $(1 - h) < (1 - \delta)$ i.e. iff $\delta < h$. Now $\max(1 + \delta - h, (1 + h)/(1 - \delta)) = \lambda_4 < 1$ and so we can find an integer n_{04} such that for all $n \geq n_{04}$, $\max(1 + \delta - c_n, (1 - c_n)/(1 - \delta)) < (1 + \lambda_4)/2$. Let

$$\max((1 - \lambda_1), (1 + \lambda_2), (1 + \lambda_3), (1 + \lambda_4)) = 2\lambda < 2 \text{ and } m = \max\{n_{01}, n_{02}, n_{03}, n_{04}\}.$$

Hence for $n \geq m$, $d(x_{n+1}, Tx_{n+1}) \leq \lambda\, d(x_n, Tx_n) \leq \cdots \leq \lambda^{n+1-m} d(x_m, Tx_m) \to 0$ as $n \to \infty$. Now $d(Tx_{n_i}, \xi) \leq d(Tx_{n_i}, x_{n_i}) + d(x_{n_i}, \xi) \to 0$ as $n_i \to \infty$. For the pair x_{n_i} and ξ at least one of (a), (b), (c) or (d) of (3) is true. Therefore at least one of these must be true infinitely often for the pair ξ, x_{n_i}, $i = 1, 2, \ldots,$. Thus for a subsequence of $\{n_i\}$, which, by relabelling we denote by $\{n_i\}$, only one of the inequalities (a), (b), (c) or (d) of (3) is true. In the case 3 (a) by putting $x = x_{n_i}$, $y = \xi$, we obtain $d(x_{n_i}, Tx_{n_i}) + d(\xi, T\xi) \leq \alpha d(x_{n_i}, \xi)$ and letting $n_i \to \infty$ we obtain $d(\xi, T\xi) \leq 0$, whence $d(\xi, T\xi) = 0$ implying $\xi = T\xi$. In the case 3 (b) we get, by putting $x = x_{n_i}$, $y = \xi$,

$$d(x_{n_i}, Tx_{n_i}) + d(\xi, T\xi) \leq \beta\{d(x_{n_i}, T\xi) + d(\xi, Tx_{n_i}) + d(x_{n_i}, \xi)\}$$

and letting $n_i \to \infty$, $d(\xi, T\xi) \le \beta d(\xi, T\xi)$, which is true only when $d(\xi, T\xi) = 0$ or $\xi = T\xi$. In the case 3 (c) we get

$$d(x_{n_i}, Tx_{n_i}) + d(\xi, T\xi) + d(Tx_{n_i}, T\xi) \le \gamma\{d(x_{n_i}, T\xi) + d(\xi, Tx_{n_i})\}$$

and using $d(x_{n_i}, T\xi) \le d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, T\xi)$ we have,

$$d(x_{n_i}, T\xi) + d(\xi, T\xi) \le \gamma\{d(x_{n_i}, T\xi) + d(\xi, Tx_{n_i})\}$$

Letting $n_i \to \infty$ we get, $2d(\xi, T\xi) \le \gamma d(\xi, T\xi)$, which is impossible unless $d(\xi, T\xi) = 0$ or $\xi = T\xi$. In the case 3 (d) we have

$$d(\xi, T\xi) \le d(\xi, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, T\xi) \le d(\xi, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) +$$
$$\delta \max\{d(x_{n_i}, \xi), d(x_{n_i}, Tx_{n_i}), d(\xi, T\xi), 1/2[d(x_{n_i}, T\xi) + d(\xi, Tx_{n_i})]\}.$$

Letting $n_i \to \infty$, we get, $d(\xi, T\xi) \le \delta \max\{d(\xi, T\xi), 1/2d(\xi, T\xi)\}$ or, $d(\xi, T\xi) \le \delta d(\xi, T\xi)$, which is impossible unless $d(\xi, T\xi) = 0$ or $\xi = T\xi$.

Pal and Maiti [5] have given some fixed point theorems for mappings T satisfying

(4) For all x, y $(x \ne y)$ at least one of the following conditions holds:

(a) $d(x, Tx) + d(y, Ty) < 2d(x, y)$,
(b) $d(x, Tx) + d(y, Ty) < 2/3 \cdot \{d(x, Ty) + d(y, Tx) + d(x, y)\}$,
(c) $d(x, Tx) + d(y, Ty) + d(Tx, Ty) < 3/2 \cdot \{d(x, Ty) + d(y, Tx)\}$.
(d) $d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$.

Our next result will show that the subsequential limit points of the sequence of Mann iterates in some of the above cases are fixed points of T.

THEOREM 3. *Let X be a Banach space, T a selfmap satisfying* (4) (*a*) *or* (*b*). *Let $\{x_n\}$ be the sequence of Mann iterates of T with $\{c_n\}$ satisfying* (*i*), (*ii*), *and* (*iv*). *Then every subsequential limit point of $\{x_n\}$ is a fixed point of T.*

Proof. Let $x_{n_i} \to \xi$ as $n_i \to \infty$. In case (4) (a) putting $x = x_n$ and $y = x_{n+1}$, we get

$$d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) < 2d(x_n, x_{n+1}) = 2c_n d(x_n, Tx_n), \text{ or,}$$
$$d(x_{n+1}, Tx_{n+1}) < (2c_n - 1)d(x_n, Tx_n).$$

Since left-hand side is to be positive (otherwise $x_{n+1} = x_{n+2} = \dots$), $c_n > 1/2$ and therefore $1 > c_n > 1/2$, or, $1 > 2c_n - 1 > 0$. In case (4)(b) is satisfied, putting $x = x_n$, $y = x_{n+1}$, we get

$$d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) < 2/3 \cdot \{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n) + d(x_n, x_{n+1})\} \le$$
$$\le 2/3\{d(x_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1}) + d(x_{n+1}, Tx_n) + d(x_n, x_{n+1})\}.$$

Using $d(x_n, x_{n+1}) = c_n d(x_n, Tx_n)$ and $d(x_{n+1}, Tx_n) = (1-c_n)d(x_n, Tx_n)$ we obtain

$$d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) < 2/3 \cdot \{(1 + c_n)\, d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\} \text{ or,}$$
$$d(x_{n+1}, Tx_{n+1}) < (2c_n - 1)d(x_n, Tx_n). \text{ As before } c_n > 1/2 \text{ and } 1 > 2c_n - 1 > 0.$$

Therefore $\{d(x_n, Tx_n)\}_{n=1}^{\infty}$ is a monotonically decreasing sequence bounded below by zero so it converges to $\alpha \geq 0$. For the pairs x_{n_i}, ξ, $i = 1, 2, 3, \dots$, at least one of the inequalities 4 (a) or 4 (b) is true. Hence at least one of these inequalities will be true for an infinite number of such pairs. In other words we can find a subseuqnce of $\{n_i\}_{i=1}^{\infty}$ which, for convenience, we relabel as $\{n_i\}_{i=1}^{\infty}$ such that either 4 (a) is true for each pair x_{n_i}, ξ or 4 (b) is true.

In case 4 (a) is true, we have, by putting $x = x_{n_i}$ and $y = \xi$, $d(x_{n_i}, Tx_{n_i}) + d(\xi, T\xi) < 2d(x_{n_i}, \xi)$. Letting $n_i \to \infty$, $\alpha + d(\xi, T\xi) \leq 0$ whence $d(\xi, T\xi) = 0$, or, $\xi = T\xi$. If the case 4 (b) holds for each pair x_{n_i}, ξ, we have, by putting $x = x_{n_i}$ and $y = \xi$,

$$\begin{aligned} d(x_{n_i}, Tx_{n_i}) + d(\xi, T\xi) &< 2/3 \cdot \{d(x_{n_i}, T\xi) + d(\xi, Tx_{n_i}) + d(\xi, x_{n_i})\} \leq \\ &\leq 2/3 \cdot \{d(x_{n_i}, T\xi) + d(\xi, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(\xi, x_{n_i})\}. \end{aligned}$$

Letting $n_i \to \infty$, $\alpha + d(\xi, T\xi) \leq 2/3 \cdot \{d(\xi, T\xi) + \alpha\}$, implying that $\alpha = 0 = d(\xi, T\xi)$. Hence $\xi = T\xi$.

The structure of the set of subsequential limit points of the sequence of iterates of the mappings satisfying (4) have been studied by Maiti and Babu [4] who have proved the following.

THEOREM 4. *Let T be a continuous selfmap of a metric space satisfying (4). Assume further that for $x \in X$, $\overline{O}(x, T)$ is compact. Than $L(x)$, the set of subsequential limit points of the sequence of iterates $\{T^n x\}$ is a nonempty, closed, compact and connected subset of $F(T)$, the set of fixed points of T. Either $L(x)$ contains exactly one point or it contains uncountably many points. In case $L(x)$ contains just one point $\lim_{m\to\infty} T^m x$ exists and belongs to $F(T)$. In case $L(x)$ is uncountable it is contained in the boundary of $F(T)$.*

This theorem is closed in spirit to the main result of Diaz and Metcalf [3]. We generalize the above theorem along the line of Theorem 3 of Rhoades [5]. Before that we shall give some definitions. Let S, T be selfmappings of a metric space X. The (T, S) orbit of a point $u \in X$ is defined as $I(u, T, S) = \{(TS)^n u \mid n = 0, 1, 2, \dots\} \cup \{S(TS)^n u \mid n = 0, 1, 2, \dots\}$. X is said to be (T, S) orbitally complete if every Cauchy sequence in $I(u, T, S)$ converges in X for all $u \in X$. $\overline{I}(u, T, S)$ will denote the closure of $I(u, T, S)$. $F(S, T)$ will denote the set of common fixed points of S and T, i. e., $F(S, T) = \{x \mid Sx = Tx\}$.

THEOREM 5. *Let T_1 and T_2 be continuous selfmaps of (X, d) and p, q fixed positive integers such that for $x \neq y$ at least one of the following is true:*

(a) $d(x, T_1^p x) + d(y, T_2^q y) < 2d(x, y)$,
(b) $d(x, T_1^p x) + d(y, T_2^q y) < 2/3 \cdot \{d(x, T_2^q y) + d(y, T_1^q x) + d(x, y)\}$,
(c) $d(x, T_1^p x) + d(y, T_2^q y) + d(T_1^p x, T_2^q y) < 3/2 \cdot \{d(x, T_2^q y) + d(y, T_1^p x)\}$,
(d) $d(T_1^p x, T_2^q y) < \max\{d(x, y), d(x, T_1^p x), d(y, T_2^q y), [d(x, T_2^q y) + d(y, T_1^p x)]/2\}$.

Assume that for $u \in X$, $\overline{I}(u, T_2^q, T_1^p)$ is compact. Let $L(x)$ be the set of subsequential limit points of the sequence $\{x_n\}_{n=0}^{\infty}$ where $x_0 = u$, $x_{2n} = (T_2^q T_1^p)^n x_0$

and $x_{2n+1} = T_p^1(T_2^q T_1^p)^n x_0$. *Then* $L(x)$ *is a nonempty, closed, compact and connected subset of* $F(T_1^p, T_2^q)$. $L(x)$ *contains either exactly one point or uncountably many points. In case* $L(x)$ *consists of just one point,* $\lim_{m\to\infty} x_m$ *exists and belongs to* $F(T_1^p, T_2^q)$. *In case* $L(x)$ *is uncountable, it is contained in the boundary of* $F(T_1^p, T_2^q)$.

Proof. The compactness of $\overline{I}(u, T_2^q, T_1^p)$ ensures the existence of subsequential limit points of $\{x_n\}$ which we shall now show to be common fixed points of T_1^p and T_2^q. Let $c_n = d(x_n, x_{n+1})$. If (a) is satisfied then $d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1})$ showing $c_{2n+1} < c_{2n}$. Putting $x = x_{2n}$ and $y = x_{2n-1}$ we get $c_{2n-1} > c_{2n}$. In case (b) is satisfied, putting $x = x_{2n}$ and $y = x_{2n+1}$, we get

$$d(x_{2n}, x_{2n+1}) + d(x_{2n+1} + x_{2n+2}) < 2/3 \cdot \{d(x_{2n}, x_{2n+2}) + d(x_{2n}, x_{2n+1})\}.$$

Using $d(x_{2n}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})$ we get $c_{2n} + c_{2n+1} < 2/3 \cdot (2c_{2n} + c_{2n+1})$, whence $c_{2n+1} < c_{2n}$. Putting $x = x_{2n}$ and $y = x_{2n-1}$ we can show as before that $c_{2n} < c_{2n-1}$. If case (c) holds, putting $x = x_{2n}$, $y = x_{2n+1}$, we get

$$\begin{aligned} d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2}) <& \\ < 3/2 \cdot \{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})\}.& \end{aligned}$$

Since $d(x_{2n}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})$ we get $c_{2n} + 2c_{2n+1} < 3/2 \cdot (c_{2n} + c_{2n+1})$ which gives $c_{2n+1} < c_{2n}$. Putting $x = x_{2n}$ and $y = x_{2n-1}$ we can similarly show that $c_{2n} < c_{2n-1}$. In case (d) is satisfied putting $x = x_{2n}$ and $y = x_{2n+1}$,

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) < \\ & < \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), [d(x_{2n}, x_{2n+2}) + \\ & \quad + d(x_{2n+1}, x_{2n+1}]/2\} \leq \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ & \qquad [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]/2\}, \end{aligned}$$

or, $c_{2n+1} < \max\{c_{2n}, c_{2n+1}, (c_{2n} + c_{2n+1})/2\}$, so that in all possible cases $c_{2n+1} < c_{2n}$. Putting $x = x_{2n}$, $y = x_{2n-1}$ we can similarly deduce that $c_{2n} < c_{2n-1}$. Thus $c_{2n+1} < c_{2n} < c_{2n-1}$ in all possible cases. Hence in all cases $\{c_n\}$ is a monotonically decreasing sequence of reals bounded below by zero and so will converge to $\alpha \geq 0$.

The compactness of $I(u, T_2^q, T_2^p)$ ensures existence of cluster, points so let $x_{n_i} \to \xi$ as $i \to \infty$. Since $\{n_i\}$ is an infinite number of integers we can choose a subsequence consisting only of odd numbers. By relabelling, if necessary, let us assume that each n_i is odd. Since each n_i is odd, we have

$$\alpha = \lim_{i\to\infty} d(x_{n_i}, x_{1+n_i}) = \lim_{i\to\infty} d(x_{n_i}, T_2^q x_{n_i}) = d(\xi, T_2^q \xi).$$

Similarly,

$$\alpha = \lim_{i\to\infty} d(x_{1+n_i}, x_{2+n_i}) = \lim_{i\to\infty} d(T_2^q x_{n_i}, T_1^p T_2^q x_{n_i}) = d(T_2^q \xi, T_1^p T_2^q \xi).$$

Furthermore

$$d(\xi, T_1^p T_2^q \xi) \leq d(\xi, T_2^q \xi) + d(T_2^q \xi, T_1^p T_2^q \xi) = 2\alpha.$$

Assuming $\xi \neq T_2^q\xi$ and putting $x = T_2^q\xi$, $y = \xi$ in (a), (b), (c) and (d), we get

in (a), $d(T_2^q\xi, T_1^pT_2^q\xi) + d(\xi, T_2^q\xi) < 2d(T_2^q\xi, \xi)$ or, $2\alpha < 2\alpha$;
in (b), $d(T_2^q\xi, T_1^pT_2^q\xi) + d(\xi, T_2^q\xi) < 2/3\{d(T_2^q\xi, T_2^q\xi) + d(\xi, T_1^pT_2^q\xi) +$
$+ d(T_2^q\xi, \xi)\}$ or $2\alpha < 2/3 \cdot 3\alpha = 2\alpha$;
in (c), $d(T_2^q\xi, T_1^pT_2^q\xi) + d(\xi, T_2^q\xi) + d(T_1^pT_2^q\xi, T_2^q\xi) < 3/2 \cdot \{d(T_2^q, T_2^q\xi) +$
$+ d(\xi, T_1^pT_2^q\xi)\}$, or $3\alpha < 3/2 - 2\alpha = 3\alpha$;

and, in (d),

$$d(T_1^pT_2^q\xi, T_2^q\xi) < \max\{d(T_2^q\xi, \xi), d(T_2^q\xi, T_1^pT_2^q\xi), d(\xi, T_2^q\xi), [d(T_2^q\xi, T_2^q\xi) + d(\xi, T_1^pT_2^q)]/2\}$$

or, $\alpha < \max\{\alpha, \alpha, \alpha, 1/2 \cdot 2\alpha\} = \alpha$.

The contradiction $\alpha < \alpha$ in all cases above shows that our assumption $\xi \neq T_2^q\xi$ is wrong. Hence $\xi = T_2^q\xi$ and $\alpha = d(\xi, T_2^q\xi) = d(T_2^q\xi, T_1^pT_2^q\xi) = 0$. Hence $T_2^q\xi = T_1^pT_2^q\xi$, or, $d = T_1^p\xi$. Thus $\xi = T_1^p\xi = T_2^q\xi$ showing that $\xi \in F(T_1^p, T_2^q)$.

Let us assume now that there exists an infinite subsequence of even integers in $(n_i)_{i=1}^{\infty}$. By relabelling, if necessary, we can assume that each n_i is even. In this case

$$\alpha = \lim_{i\to\infty} d(x_{n_i}, x_{1+n_i}) = \lim_{i\to\infty} d(x_{n_i}, T_1^p x_{n_i}) = d(\xi, T_1^p\xi)$$

and

$$\alpha = \lim_{i\to\infty} d(x_{1+n_i}, x_{2+n_i}) = \lim_{i\to\infty} d(T_1^p x_{n_i}, T_2^qT_1^p x_{n_i}) = d(T_1^p\xi, T_2^qT_1^p\xi).$$

Further $d(\xi, T_2^qT_1^p\xi) \leq d(\xi, T_1^p\xi) + d(T_1^p\xi, T_2^qT_1^p\xi) = 2\alpha$. Putting $x = \xi$, $y = T_1^p\xi$ in the inequalities (a), (b), (c) and (d), we can derive that the assumption $\xi \neq T_1^p\xi$ leads to $\alpha > \alpha$ proving thereby that $\xi = T_1^p\xi$. Hence $d(T_1^p\xi, T_2^qT_1^p\xi) = d(T_1^p\xi, \xi) = 0$ and so $T_1^p\xi = T_2^qT_1^p\xi$, or, $\xi = T_2^q\xi$. Thus $\xi = T_1^p\xi = T_2^q\xi$. Hence $\xi \in F(T_1^p, T_2^q)$.

We have shown, therefore, that all cluster points of $\{x_n\}_{n=1}^{\infty}$ are common fixed points of T_1^p and T_2^q. In other words $L(x) \subset F(T_1^p, T_2^q)$.

Since we have assumed $I(x_0, T_2^q, T_1^p)$ to be compact, we have $\emptyset \neq L(x) \subset F(T_1^p, T_2^q)$, Since $L(x)$ is a closed subset of $I(x_0, T_2^q, T_1^p)$, which is compact we conclude that $L(x)$ is compact. Thus $L(x)$ is a closed and compact subset of $F(T_1^p, T_2^q)$.

To prove that $L(x)$ is connected we assume the contrary. Hence there exists a pair of nonempty, disjoint, closed subsets S_1 and S_2 of $L(x)$ such that $S_1 \cup S_2 = L(x)$. Since S_1 and S_2 are closed subsets of the compact set $L(x)$, they are themselves compact. Hence $d(S_1, S_2) > 0$. We have shown in the course of the proof that $\alpha = \lim_{i\to\infty} d(x_n, x_{n+1}) = 0$. We now proceed to show that $d(x_n, L(x)) \to 0$ as $n \to \infty$. For, if not, then for some $\varepsilon > 0$ there exists a subsequence $\{n_i\}_{i=1}^{\infty}$ such that $d(x_{n_i}, L(x)) \geq \varepsilon > 0$. Since $\overline{I}(x_0, T_2^q, T_1^p)$ is compact, the sequence $\{x_{n_i}\}_{i=1}^{\infty}$, has a subsequence $\{x_{m_i}\}_{i=1}^{\infty}$ say, converging to $\xi \in L(x)$. Therefore $d(x_{m_i}, L(x)) \leq d(x_{m_i}, \xi) \to 0$ as $i \to \infty$, contradicting our hypothesis. Hence $d(x_n, L(x)) \to 0$ as $n \to \infty$. Thus we can find an integer M such that for $m \geq M$, $d(x_m, x_{m+1}) < d(S_1, S_2)/3$ and $d(x_m, L(x)) < d(S_1, S_2)/3$. Given $m \geq M$, there exists $s \in L(x) = S \cup S_2$ such that $d(x_m, S_1 \cup S_2) = d(x_m, s)$, since $L(x)$ is compact. If $s \in S_1$, then $d(x_m, S_1) \leq d(x_m, s) < d(S_1, S_2)/3$. Therefore, for any $m \geq M$, either $d(x_m, S_1) < d(S_1, S_2)/3$, or, $d(x_m, S_2) < d(S_1, S_2)/3$. But both these inequalities cannot hold simultaneously, because in that case $d(S_1, S_2) \leq d(S_1, x_m) + d(x_m, S_2) < 2/3 \cdot d(S_1, S_2)$, which is absurd. Next we see that the set of positive integers $m \geq M$ for which $d(x_m, S_1) < d(S_1, S_2)/3$ is nonempty since $\emptyset \neq S_1 \subset L(x)$. Similarly the set of positive integers $m \geq M$ for which $d(x_m, S_2) < d(S_1, S_2)/3$ is nonempty. Suppose that for $m_1 > M$, $d(x_{m_i}, S_1) < d(S_1, S_2)/3$. Then there exist integers $n > m_1$ such that $d(x_n, S_2) < d(S_1, S_2)/3$. Let $k+1$ be the smallest of them. Then $d(x_k, S_1) < d(S_1, S_2)/3$ and $d(x_{k+1}, S_2) < d(S_1, S_2)/3$. Now, one has

$$\begin{aligned} d(S_1, S_2) &\leq d(S_1, x_k) + d(x_k, x_{k+1}) + d(x_{k+1}, S_2) < \\ &< d(S_1, S_2)/3 + d(S_1, S_2)/3 + d(S_1, S_2)/3 = d(S_1, S_2). \end{aligned}$$

This is absurd. Therefore the assumption that $L(x) = S_1 \cup S_2$ with S_1 and S_2 nonempty, disjoint and closed is false. In other words $L(x)$ is connected.

Since $L(x)$ is connected it is either a singleton or is uncountable. In case $L(x) = \{\xi\}$, we have $d(x_m, \xi) = d(x_m, L(x)) \to 0$, as $m \to \infty$, showing that $x_m \to \xi$. In case $L(x)$ is uncountable and $\xi \in L(x)$ is an interior point of $F(T_1^p, T_2^q)$, then $F(T_1^p, T_2^q)$ must contain an element x_m in its interior. Hence $T_1^p x_m = x_m = T_2^q x_m$. Thus $x_m = x_{m+1} = x_{m+2} = \dots$. This reduces $L(x)$ to a singleton contrary to hypothesis. Therefore, in this case $L(x)$ is contained in the boundary of $F(T_1^p, T_2^q)$.

Remark. We can see that similar results hold also for the sequence $\{x_n\}$, $x_{2n} = (T_1^p, T_2^q)^n x_0$, $x_{2n+1} = T_2^q (T_1^p, T_2^q)^n x_0$.

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