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COMPLEX HYPERSURFACES OF A GENERALIZED HOPF MANIFOLD

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Abstract. We study complex hypersurfaces of a generalized Hopf manifold (g.H.m.) using the second fundamental form and structure equations. When the ambient manifold is conformally-flat ($\mathcal{P}_0K$-manifold) we obtain some results about the curvature of complex submanifolds and their stability with respect to normal variations.

1. Introduction. In [7] Vaisman considered a generalisation of Hopf manifolds presenting a special interest for differential geometry. Let $(\overline{M}, J)$ be a complex manifold of complex dimension n with complex structure tensor J; it is said to be locally conformal Kähler (l.c.K) if it bears a hermitian metric g conformally related with a Kähler metric in a certain neighbourhood of each point x of $\overline{M}$. An equivalent definition-more suitable for applications-requires the existence of a closed globally defined 1-form ω (the Lee form) on M related to the fundamental 2-form Ω of $\overline{M}$ by the equation

$$d\Omega = \omega \wedge \Omega. \tag{1.1}$$

When $\omega = 0$ on $\overline{M}$ (resp. ω is exact) the manifold is Kähler (resp. globally conformal Kähler). We shall not consider these cases here.

L.c.K manifolds appear naturally in Gray-Hervella classification in the class W_4.

A generalized Hopf manifold (g.H.m.) is a l.c.K. manifold whose Lee form is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of g (i.e. $\omega\overline{\nabla} = 0$).

It is known that Hopf manifolds are l.c.K. and their Lee form is parallel. In addition they cannot have Kähler metric and this is one point, among others, that motivates the study of g.H.m. We shall suppose ω without singularities, i.e. $\omega \neq 0$ everywhere (the manifold is then called strongly non-Kähler).

Let $u = \omega/|\omega|$ be the corresponding unitary one-form and U the associated unit vector field, i.e. $g(U, X) = u(X)$ (We call it the structure vector field, or, also,

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the Lee vector field). Let us put $V = JU$ and $v(X) = g(V, X)$ for every tangent vector X. It is not difficult to show that a l.c.K manifold is g.H.m. iff $C = |\omega|/2$ is constant on $\overline{M}$ (every manifold is supposed connected) and $\overline{\nabla} u = 0$.

On a l.c.K. manifold one has to consider the Weyl connection $\overline{\overline{\nabla}}$ which is just the Levi-Civita connection of the locally conformal (almost) Kähler metrics and is given by:

$$\overline{\overline{\nabla}}_X Y = \overline{\nabla}_X Y - C[u(X)Y + u(Y)X - g(X,Y)U] \tag{1.2}$$

Obviously $\overline{\overline{\nabla}} J = 0$ [6]. Now, a g.H.m. on which the curvature tensor $\overline{\overline{R}}$ of the Weyl connection is identically zero is called a $\mathcal{P}_0 K$-manifold.

2. Basic formulas. In the sequel we shall consider complex hypersurfaces isometrically immersed in l.c.K. (and g.H.m.) manifolds and tangent to the structure vector field U. We call them invariant hypersurfaces. We denote by the same letter the induced metric tensor g and complex structure J the restrictions of ω and Ω. Let ∇ be the induced Levi-Civita connection and b the second fundamental tensor. Now let $\overline{M}$ be a g.H.m and M an invariant hypersurface. Obviously one has $d\Omega = \omega \wedge \Omega$ on M, so (M, J, g) is l.c.K. Using Gauss formula for the hypersurface M one shows that ω is parallel on M. Indeed, for any X, Y tangent to M we have:

$$(\nabla_X \omega)Y = \nabla_X(\omega(Y)) - \omega(\nabla_X Y) = \overline{\nabla}_X(\omega) - \omega(\overline{\nabla}_X Y) + \omega(b(X,Y)) = 0 \tag{2.1}$$

so we can state

PROPOSITION 2.1 . *An invariant hypersurface M of a l.c.K manifold $\overline{M}$ is also l.c.K; if $\overline{M}$ is g.H.m, so is M.*

Let h and k be the second fundamental forms associated with the normal sections N and JN, A and A' the corresponding Weingarten operators. Using the condition $\overline{\overline{\nabla}} J = 0$ and (1.2), and separating the terms which are tangent to M and those which are normal one derives:

PROPOSITION 2.2. *For an invariant hypersurface of al l.c.K. manifold, the following formulas are valid:*

$$(\nabla_X J)Y = C[u(Y)JX - u(JY)X - g(X,Y)JU + g(X,JY)U] \tag{2.2}$$

$$h(X,Y) = k(X,JY); \qquad k(X,Y) = -h(X,JY) \tag{2.3}$$

On the other side, from the Weingarten formula written for A and A' one shows

LEMMA 2.3. a) *$JA = -AJ$; $A' = JA$;* b) *A and A' are symmetric operators with respect to the scalar product induced by the metric g on the tangent bundle TM of M.*

Using this we get

LEMMA 2.4. (Smyth [4]) *There exists an orthonormal basis* $\{E_i, JE_i\}$, $1 \leq i \leq n-1$ *in each tangent space of the complex hypersurface* M *with respect to which the Weingarten operator* A *is diagonal of the form*

$$\begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_{n-1} & & & \\ & & & -\lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & -\lambda_{n-1} \end{bmatrix}. \tag{2.4}$$

COROLLARY 2.5. $Tr A = Tr A' = 0$, *so* M *is minimal.*

Remark 2.6. It is true in general that a submanifold tangent to the Lee vector of a l.c.K manifold is minimal. This can be proved either directly or by using Theorem 5.1 of [7].

Now, if $\overline{M}$ is g.H.m. we have:

$$\nabla_X U = \overline{\nabla}_X U - b(X,U) = -h(X,U)\,N - k(X,U)\,JN, \tag{2.5}$$

$$\nabla_X U = 0, \quad h(X,U) = 0, \quad k(X,U) = 0. \tag{2.6}$$

Therefore $g(AU, X) = g(JAU, K) = 0$ so

$$AU = AV = 0 \tag{2.7}$$

From (2.2) we have

$$\nabla_X V = C[JX + v(X) - u(X)V] \tag{2.8}$$

for every tangent vector X.

We conclude this section with some results about $\mathcal{P}_0K$-manifolds. First, one has the formula:

$$\begin{aligned}\overline{R}(X,Y)Z = C^2[&u(X)u(Z)Y - u(Y)u(Z)X - u(X)g(Y,Z)U+ \\ &+ u(Y)g(X,Z)U + g(Y,Z)X - g(X,Z)Y]\end{aligned} \tag{2.9}$$

for any X, Y, Z tangent to $\overline{M}$ [6].

It is worth mentioning that on the orthogonal distribution to the Lee vector field U, the curvature tensor $\overline{R}$ has the form of the curvature tensor of a space of constant curvature.

From a direct calculation one has:

PROPOSITION 2.7. *A* $\mathcal{P}_0K$*-manilfold is a locally symmetric manifold.*

The Ricci tensor and the scalar curvature of a $\mathcal{P}_0K$-manifold are given by the formulas

$$\overline{S}(X,Y) = 2C^2(n-1)[g(X,Y) - u(X)u(Y)], \tag{2.10}$$

$$\overline{r} = 2C^2(n-1)(2n-1). \tag{2.11}$$

At the end of this paragraph, let us indicate a way to construct examples of invariant hypersurfaces. If the vector fields U and V give a regular foliation $\mathcal{F}$ on a compact g.H.m $\overline{M}$, the leaf space $\overline{N} = \overline{M}/\mathcal{F}$ is a compact Kähler manifold and the projection $\pi : \overline{M} \to \overline{N}$ is locally a trivial fiber budle [2]. Let N be a complex hypersurface of $\overline{N}$. Then $M = \pi^{-1}(N)$ is a complex hypersurface of $\overline{M}$ and U, V are both tangent to M.

3. Induced f-structure. It is known that a l.c.K. manifold carries a natural f-structure f (i.e. a $(1,1)$-tensor field satisfying $f^3 + f = 0$, see [8]) given by $f = J + v \otimes U - u \otimes V$. This f-structure has two complemented frames defined by the vector fields U and V. Moreover, if $\overline{M}$ is g.H.m., the f-structure is normal i.e.

$$N_f(X,Y) + du\,(X,Y)U + dv\,(X,Y)V = 0 \tag{3.1}$$

where N_f is the Nijenhuis tensor f [6]. Obviously, a complex hypersurface tangent to U inherits an f-structure. Taking the restriction of (3.1) to M we obtain

Proposition 3.1. *An invariant complex hypersurface of a g.H.m. is a normal f-manifold with two complemented frames.*

4. Curvature. In the present section we shall study properties of the curvature operators of an invariant hypersurface.

First of all, from Gauss formula and Lemma 2.3 we get:

$$\begin{aligned} R(X,Y;Z,W) = \overline{R}(X,Y;Z,W) + \{g(AX,Z)g(AY,W) - g(AX,W)\\ + g(AY,Z)\} + \{g(JAX,X)g(JAY,Y) - g(JAX,W)g(JAY,Z)\} \end{aligned} \tag{4.1}$$

for every X, Y, Z, W tangent to an invariant hypersurface of a l.c.K manifold.

If P is 2-plane in T_xM, $x \in M$, then its sectional curvature is given by:

$$\begin{aligned} K(P) = \overline{K}(P) + \{g(AX,X)g(AX,Y) - g(AX,Y)^2\} -\\ - \{g(JAX,X)g(JAY,Y) - g(JAY,Y)^2\}, \end{aligned} \tag{4.2}$$

where X, Y is an orthonormal basis of P.

Corollary 4.1. *If X is a unit vector tangent to M, at x, the holomorphic sectional curvature of $\overline{M}$ and M at x are related by:*

$$K(X) = \overline{K}(X) - 2\{g(AX,X)^2 + g(JAX,X)^2\} \tag{4.3}$$

In the sequel, $\overline{M}$ will be a $\mathcal{P}_0K$-manifold. The Ricci tensor and the scalar curvature of an invariant hypersurface are respectively:

$$S(X,Y) = 2C^2(n-2)[g(X,Y) - u(X)u(Y)] - 2g(AX,AY) \tag{4.4}$$

$$r = 2C^2(n-2)(2n-3) - 2\mathrm{Tr}\,(A^2) \tag{4.5}$$

From (4.4) we easily derive:

COROLLARY 4.2. *There are no Einstein invariant hypersurfaces in a $\mathcal{P}_0K$-manifold.*

The holomorphic sectional curvature of a 2-plane $\{X, JX\}$ is given by:

$$K(X) = C^2[1 - u(X)^2 - u(JX)^2] - 2[g(AX,X)^2 + g(JAX,X)^2]. \tag{4.6}$$

Now, let N be a unit normal section to M. From (2.9) one obtains:

$$\overline{R}(X,Y)N = 0 \tag{4.7}$$

for any X, Y tangent to M. The curvature tensor in the normal bundle $TM^{\perp}$ of M is defined by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N, \tag{4.8}$$

where $\nabla^{\perp}$ is the normal connection induced in the normal bundle. We put $[A, A'] = AA' - A'A$. Then Codazzi equation takes the form:

$$(\overline{R}(X,Y)N)^{\perp} = R^{\perp}(X,Y)N - g([A,A']X,Y). \tag{4.9}$$

From (4.7) we see that $R^{\perp} = 0$ iff

$$g([A,A']X,Y) = 0. \tag{4.10}$$

But $A' = JA$ so (4.10) is equivalent to

$$g(A^2JX,Y) + g(JA^2X,Y) = 0. \tag{4.11}$$

From this and $JA = -AJ$ we get $A = 0$, hence:

THEOREM 4.3. *Let M be a $\mathcal{P}_0K$-manifold and M an invariant hypersurface. Then M is totally geodesic if and only if its normal bundle is flat.*

Since a $\mathcal{P}_0K$-manifold is a locally symmetric space a natural question to ask is whether an invariant hypersurface inherits this property.

By a straightforward calculation we obtain

THEOREM 4.4. *Let M be an invariant hypersurface of a $\mathcal{P}_0K$-manifold M. If the second fundamental form of M is parallel, then M is locally symmetric.*

COROLLARY 4.5. *A totally geodesic invariant hypersurface of $\mathcal{P}_0K$-manifold is locally symmetric.*

One can weaken the condition of local symmetry and impose, instead, the algebraic condition

$$R(X,Y) \circ R = 0 \tag{4.12}$$

for all tangent vector X and Y, where the endomorphism $R(X,Y)$ operates on R as a derivation of the tensor algebra at each point of M (Cartan [1]). Szabó called the spaces satisfying this condition semi-symmetric and classified them in [5]. Moreover, he constructed many examples of semi-symmetric, nonlocally symmetric spaces.

Now we consider an orthonormal basis $\{E_i, JE_i\}$, $i = \overline{1, n-1}$ where $E_1 = U$ and $JE_1 = V$. Writing (4.12) in extenso for all possible combinations of vectors from this basis (i.e. $R(E_i, E_j)R(E_k, E_l)E_m = 0$) after a long calculation we conclude that all the eigenvalues of A must be zero, otherwise the resulting equations are contradictory. Hence we can state the following:

THEOREM 4.6. *An invariant hypersurface of a $\mathcal{P}_0K$-manifold is semi-symmetric if and only if it is totally geodesic.*

Hypersurfaces with recurrent normal section. A tangent vector X is said to be recurrent with respect to a covariant derivative ∇ if there exists a one form α so that $\nabla_Y X = \alpha(Y)X$ for every tangent vector Y. In this paragraph we suppose the normal section N is recurrent in the normal bundle of an invariant complex hypersurface, so that we have

$$\nabla^{\perp}_X N = \alpha(X)N \tag{5.1}$$

for a one-form α and for every tangent vector X. We obtain immediately

LEMMA 5.1. *If N is recurrent, then we have*

$$R^{\perp}(X,Y)N = d\alpha(X,Y)N \tag{5.2}$$

Combined with Theorem 4.1 this yields to:

COROLLARY 5.2. *If N is recurrent and α is closed, then M is totally geodesic. In particular this is true if $\nabla^{\perp}_X N = u(X)N$.*

In the sequel, M will be compact. We recall that the normal variation induced from a normal vector field N of a compact minimal submanifold M is defined by

$$\vartheta(N) = \int_M \{||\nabla^{\perp} N||^2 - \overline{S}(N) - ||A_N||^2\}\, dv \tag{5.3}$$

where $\overline{S}(N) = \sum_{i=1}^{2n-2} \overline{R}(N, E_i; E_i, N)$ and $\nabla^{\perp} N$ is regarded as a (1,1) tensor field on the tangent bundle TM of M (here $E_1, E_1, \ldots, E_{2k-2}$ is an orthonormal basis of T_xM).

We recall also that a minimal submanifold is called stable if $\vartheta''(N) > 0$ for any normal vector field N, otherwise M is said to be unstable. We shall calculate the second variation induced from a recurrent normal section of an invariant hypersurface of a $\mathcal{P}_0K$-manifold.

From (2.10) we have

$$\overline{S}(N) = C^2(n-1). \tag{5.4}$$

Then $||A_N||^2 = \sum_{i=1}^{2n-2} g(A_N E_i, A_N E_i) = \sum_{i=1}^{2n-2} g(A_N^2 E_i, E_i) = \mathrm{Tr}\, A_N^2$.

Hence, using (4.5) we get $||A_N||^2 = C^2(n-2)(2n-3) - r/2$. If N is recurrent then

$$||\nabla^\perp N||^2 = \sum_{i=1}^{2n-2} g(\nabla^\perp_{E_i} N, \nabla^\perp_{E_i} N) = |\alpha|^2.$$

Whence we obtain

$$\begin{aligned}\vartheta''(N) &= \int_M \{|\alpha|^2 - 2C^2(n-1) - C^2(n-2)(2n-3) + r/2\} dv \\ &= \int_M \{|\alpha|^2 - C^2(2n^2 - 5n + 4) + r/2\} dv\end{aligned}$$

so we can state the following:

THEOREM 5.3. *If one of the following conditions holds:* (1) N *is parallel and* $r \leq 0$; (2) N *is recurrent with* $\alpha = u$ *and* $r \leq 0$, *then the hypersurface is unstable.*

Recently, the second author studied some submanifolds of l.c.k. manifolds [3].

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