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SOME SPECIAL CASES OF PARALLEL DISPLACEMENTS IN RECURRENT FINSLER SPACES

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Abstract. Some special cycles of line elements in the recurrent Finsler space F_n are considered. If the vector is parallely transported along one of the cycles of lineelements the difference between the original vector and the one obtained after parallel transportation is expressed by some of the curvature tensor. The method used here is the generalisation of that, used by Varga [1], for the non-recurrent Finsler space.

1. Introduction. Let us consider Finsler space F_n in which the metric function is $F(x, \dot{x})$ and the metric tensor is defined by

$$g_{\alpha\beta}(x,\dot{x}) = 2^{-1}\dot{\partial}_\alpha \dot{\partial}_\beta F^2(x,\dot{x}).$$

Definition 1.1. The Finsler space is called recurrent and is denoted by $\overline{F}_n$ when there exist vector fields $\lambda_\gamma(x,\dot{x})$ and $\mu_\gamma(x,\dot{x})$ homogeneous of degree zero in $\dot{x}$ such that [2]

$$g_{\alpha\beta}|_\gamma = \partial_\gamma\, g_{\alpha\beta} - F\dot{\partial}_\delta\, g_{\alpha\beta}\, \Gamma^{*\ \delta}_{\circ\ \gamma} - \Gamma^{*\delta}_{\alpha\ \gamma}\, g_{\delta\beta} - \Gamma^{*\delta}_{\beta\ \gamma}\, g_{\alpha\delta} = \lambda_\gamma\, g_{\alpha\beta} \tag{1.1}$$

$$g_{\alpha\beta}|_\gamma = F\dot{\partial}_\delta\, g_{\alpha\beta}\,(\delta^\delta_\gamma - A_{\circ\ \gamma}^{\ \delta}) - A_{\alpha\ \gamma}^{\ \delta}\, g_{\alpha\beta} - A_{\beta\ \gamma}^{\ \delta}\, g_{\delta\beta} = \mu_\gamma\, g_{\alpha\beta} \tag{1.2}$$

$$D\, g_{\alpha\beta} = g_{\alpha\beta}|_\gamma dx^\gamma + g_{\alpha\beta}|_\gamma Dl^\gamma \tag{1.3}$$

$$Dl^\gamma = dl^\gamma + \Gamma^{*\gamma}_{\circ\ \beta}\, dx^\beta + A_{\circ\ \beta}^{\ \gamma}\, Dl^\beta, \tag{1.4}$$

where D denotes the absolute differential which corresponds to the change of the lineelement from $(x,\dot{x})$ to $(x+dx, \dot{x}+d\dot{x})$ and "$\circ$" means the contracton by l. The connection coefficients Γ^* and A are determined under conditions

$$\Gamma^*_{\alpha\beta\gamma} = \Gamma^*_{\gamma\beta\alpha} \tag{1.5}$$

$$A_{\alpha\beta\gamma} = A_{\gamma\beta\alpha}. \tag{1.6}$$

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From (1.1) and (1.5) $\Gamma^*_{\alpha\beta\gamma}$ may be determined in the unique way and similarly (1.2) and (1.6) determine $A_{\alpha\beta\gamma}$. The connection coefficients obtained in this way are generalisations of the Cartan connections in the case of a non recurrent Finsler space (when $\lambda_\gamma = 0$ and $\mu_\gamma = 0$).

Using the notation $\{T_{\gamma\alpha\beta}\} + \{\gamma\alpha\beta\} = T_{\gamma\alpha\beta} + T_{\alpha\beta\gamma} - T_{\beta\gamma\alpha}$ we have [3]

$$2\,\Gamma^*_{\alpha\beta\gamma} = \{\partial_\gamma\, g_{\alpha\beta} - F\dot{\partial}_\delta\, g_{\alpha\beta}\,\Gamma^{*\delta}_{\circ\ \gamma} - \lambda_\gamma\, g_{\alpha\beta}\} + \{\gamma\alpha\beta\} \tag{1.7}$$

$$2\,\Gamma^*_{\circ\beta\gamma} = 2\,\gamma_{\alpha\beta\gamma}\, l^\alpha - F\dot{\partial}_\delta\, g_{\beta\gamma}\Gamma^{*\delta}_{\circ\ \circ} - (\lambda_\gamma l_\beta + \lambda_0 g_{\beta\gamma} - \lambda_\beta l_\gamma) \tag{1.8}$$

$$2\,\Gamma^*_{\circ\ \beta\circ} = 2\,\gamma_{\circ\ \beta\circ} - (2\lambda_o l_\beta - \lambda_\beta), \tag{1.9}$$

where $\gamma_{\alpha\beta\gamma}$ is the Christoffel symbol. Further we obtain

$$2\,A_{\alpha\beta\gamma} = \{F\dot{\partial}_\alpha\, g_{\beta\gamma} - F\dot{\partial}_\delta\, g_{\beta\gamma}\, A_\circ{}^\delta{}_\alpha - \mu_\alpha\, g_{\beta\gamma}\} + \{\alpha\beta\gamma\} \tag{1.10}$$

$$2\,A_{\circ\beta\gamma} = -F\dot{\partial}_\delta\, g_{\beta\gamma} A_\circ{}^\delta{}_\circ - (\mu_\circ g_{\beta\gamma} + \mu_\gamma l_\beta - \mu_\beta l_\gamma) \tag{1.11}$$

$$2\,A_{\circ\ \beta\circ} = -(2\mu_\circ l_\beta - \mu_\beta) \tag{1.12}$$

We shall suppose that in $\overline{F}_n$ all vector and tensor fields are homogeneous of degree zero in $\dot{x}$.

Lemma 1.1. *If in* $\overline{F}_n \xi^\alpha{}_{|\beta}$ *and* $\xi^\alpha|_\beta$ *are defined by*

$$\xi^\alpha{}_{|\beta} = \partial_\beta \xi^\alpha - F\dot{\partial}_\delta\, \xi^\alpha\, \Gamma^{*\delta}_{\circ\ \beta} + \Gamma^{*\alpha}_{\delta\ \beta}\xi^\delta \tag{1.13}$$

$$\xi^\alpha|_\beta = F\dot{\partial}_\delta\, \xi^\alpha(\delta^\delta_\beta - A_\circ{}^\delta{}_\beta) + A_\delta{}^\alpha{}_\beta \xi^\delta, \tag{1.14}$$

then

$$\xi_{\alpha|\beta} = \partial_\beta \xi_\alpha - F\dot{\partial}_\alpha\, \xi_\alpha\, \Gamma^{*\delta}_{\circ\ \beta} - \Gamma^{*\delta}_{\alpha\ \beta}\xi_\delta \tag{1.15}$$

$$\xi_\alpha|_\beta = F\dot{\partial}_\delta\, \xi_\alpha(\delta^\delta_\beta - A_\circ{}^\delta{}_\beta) - A_\alpha{}^\delta{}_\beta \xi_\delta \tag{1.16}$$

Proof. From $\xi_{\alpha|\beta} = (g_{\alpha\delta}\xi^\delta)_{|\beta} = g_{\alpha\delta|\beta}\xi^\delta + g_{\alpha\delta}\xi^\delta_{|\beta}$ by using (1.13) (1,1) and

$$g_{\alpha\delta}\partial_\beta \xi^\delta = \partial_\beta \xi_\alpha - \xi^\delta \partial_\beta g_\beta g_{\alpha\delta}$$

$$g_{\alpha\delta}\dot{\partial}_\chi \xi^\delta = \dot{\partial}_\chi \xi_\alpha - \xi^\delta \dot{\partial}_\chi g_{\alpha\delta} \tag{1.17}$$

we obtain (1.15). From

$$\xi_\alpha|_\beta = (g_{\alpha\delta}\xi^\delta)|_\beta = g_{\alpha\delta}|_\beta \xi^\delta + g_{\alpha\beta}\xi^\delta|\ _\beta$$

by using (1.16) (1.2) and (1.17) we have (1.16).

Using the notations of (1.13)–(1.16) we have

$$D\xi^{\alpha} = \xi^{\alpha}{}_{|\beta}\, dx^{\beta} + \xi^{\alpha}|_{\beta} Dl^{\beta}, \quad D\xi_{\alpha} = \xi_{\alpha|\beta}\, dx^{\beta} + \xi_{\alpha|\beta} Dl^{\beta}.$$

Lemma 1.2. *In $\overline{F}^{n}$ vector dx is normal to λ iff $\mu + 2l$ is normal to Dl i.e.*

$$\lambda_{\gamma}\, dx^{\gamma} = 0 \Leftrightarrow (\mu_{\gamma} + 2\, l_{\gamma})\, Dl^{\gamma} = 0.$$

Proof. From $g_{\alpha\beta} l^{\alpha} l^{\beta} = 1$ we get $Dg_{\alpha\beta} l^{\alpha} l^{\beta} + g_{\alpha\beta} l^{\alpha} Dl^{\beta} = 0$.

Using (1.3), (1.1) and (1.2) we have

$$\lambda_{\gamma}\, dx^{\gamma} = 0 \Leftrightarrow (\mu_{\gamma} + 2\, l_{\gamma})\, Dl^{\gamma} = 0. \tag{1.18}$$

from which the statement follows.

An obvious consequence of (1.18) is:

Lemma 1.3. *If the vector l is parallely transported from $(x, \dot{x})$ to $(x + dx, \dot{x} + d\dot{x})$ i.e. $Dl^{\gamma} = 0$ then $\lambda_{\gamma}\, dx^{\gamma} = 0$, which means that dx is normal to λ.*

For any vector field $\xi^{\alpha}(x, \dot{x})$ we have

$$D\, \xi^{\alpha} = d\xi^{\alpha} + w^{\alpha}_{\beta}(d) \xi^{\beta} \tag{1.19}$$

where

$$w^{\alpha}_{\beta}(d) = \Gamma^{*\alpha}_{\beta\ \ \gamma} dx^{\gamma} + A_{\beta}{}^{\alpha}{}_{\gamma} Dl^{\gamma} \tag{1.20}$$

From (1.4) we obtain

$$Dl^{\delta} I^{\gamma}_{\delta} = dl^{\gamma} + \Gamma^{*\gamma}_{\circ\ \ \beta} dx^{\beta}, \tag{1.21}$$

where $I^{\gamma}_{\delta} = \delta^{\gamma}_{\delta} - A_{\circ}{}^{\gamma}{}_{\delta}$.

Let us suppose that $[I^{\gamma}_{\delta}]$ is a regular matrix whose inverse is $[J^{\theta}_{\gamma}]$

$$I^{\gamma}_{\delta} J^{\theta}_{\gamma} = \delta^{\theta}_{\delta} \tag{1.22}$$

From (1.21) it follows $Dl^{\theta} = (dl^{\chi} + \Gamma^{*\chi}_{\circ\ \ \gamma} dx^{\gamma}) J^{\theta}_{\chi}$.

Further from $l^{\chi} = F^{-1} \dot{x}^{\chi}$ and

$$dl^{\chi} = (\partial_{\gamma} F^{-1} dx^{\gamma} - F^{-2} l_{\gamma} d\dot{x}^{\gamma}) x^{\chi} + F^{-1} \dot{x}^{\chi} \tag{1.23}$$

we have

$$Dl^{\theta} = J^{\theta}_{\chi} [(\Gamma^{*\chi}_{\circ\ \ \gamma} - F^{-1} l^{\chi} \partial_{\gamma} F) dx^{\gamma} + (\delta^{\chi}_{\gamma} - l_{\gamma} l^{\chi}) d\dot{x}^{\gamma}].$$

2. Connection coefficients Γ and C. $w^{\alpha}_{\beta}(d)$ appearing in (1.19) and (1.20) may be written in the form

$$w^{\alpha}_{\beta}(d) = \Gamma_{\beta}{}^{\alpha}{}_{\gamma} dx^{\gamma} + C_{\beta}{}^{\alpha}{}_{\gamma} d\dot{x}^{\gamma}. \tag{2.1}$$

The connection coefficients Γ^* and A from (1.20) are uniquely determined under conditions (1.1), (1.2), (1.5) and (1.6). They are given by (1.7)–(1.12). We are going to obtain relations between Γ, C and Γ^* and A. For that reason we shall equate the right hand side of (1.20) and (2.1) and use the relations (1.18), (1.23) and obtain

$$\Gamma_{\beta\ \gamma}^{\ \alpha} dx^\gamma + C^\alpha_{\beta\gamma} d\dot{x}^\gamma = \Gamma^{*\alpha}_{\beta\gamma} dx^\gamma + A_{\beta\ \gamma}^{\ \alpha} Dl^\gamma + \theta^\alpha_\beta [\lambda_\gamma dx^\gamma + (\mu_\gamma + 2l^\gamma) Dl^\gamma]$$

or

$$\begin{aligned} \Gamma_{\beta\ \gamma}^{\ \alpha} dx^\gamma + C^\alpha_{\beta\gamma} d\dot{x}^\gamma &= (\Gamma^{*\chi}_{\beta\gamma} + \theta^\alpha_\beta \lambda_\gamma) dx^\gamma \\ [A^\alpha_{\beta\theta} + \theta^\alpha_\beta(\mu_\theta + 2l_\theta)] J^\theta_\chi [(\Gamma^{*\chi}_{\circ\ \gamma} &- F^{-1} l^\chi \partial_\gamma F) dx^\gamma + F^{-1}(\delta^\chi_\gamma - l_\gamma l^\chi) d\dot{x}^\gamma], \end{aligned} \tag{2.2}$$

where $\theta^\alpha_\beta = \theta^\alpha_\beta(x, \dot{x})$ is any tensor homogeneous of degree zero in $\dot{x}$. By equating the coefficients becide dx^γ and $d\dot{x}^\gamma$ we obtain

$$\Gamma^\alpha_{\beta\gamma} = \gamma^{*\alpha}_{\beta\gamma} + \theta^\alpha_\beta \lambda_\gamma + [A^\alpha_{\gamma\theta} + \theta^\alpha_\beta(\mu_\theta + 2l_\theta)] J^\theta_\chi (\Gamma^{*\chi}_{\circ\ \gamma} - F^{-1} l^\chi \partial_\gamma F), \tag{2.3}$$

$$C_{\beta\ \gamma}^{\ \alpha} = [A_{\beta\ \theta}^{\ \alpha} + \theta^\alpha_\beta(\mu_\theta + 2l_\theta)] J^\theta_\chi F^{-1}(\delta^\chi_\gamma - l_\gamma l^\chi) \tag{2.4}$$

Lemma 2.1. *The relation*

$$C^\alpha_{\beta\gamma} \dot{x}^\gamma = F C^\alpha_{\beta\circ} = 0 \tag{2.5}$$

is valid for any θ^α_β.

The proof is obvious from (2.4).

For $\theta^\alpha_\beta = 0$, (2.3) and (2.4) become [4]

$$\Gamma_{\beta\ \gamma}^{\ \alpha} = \Gamma^{*\alpha}_{\beta\ \gamma} + A^\alpha_{\beta\theta} J^\theta_\chi (\Gamma^{*\chi}_{\circ\ \gamma} - F^{-1} l^\chi \partial_\gamma F) \tag{2.6}$$

$$C^\alpha_{\beta\gamma} = A^\alpha_{\beta\theta} J^\theta_\chi F^{-1} (\delta^\chi_\gamma - l_\gamma l^\chi) \tag{2.7}$$

Formulae (2.6) and (2.7) are not practical for calculation because they contain the term J^θ_χ, for which all we know is the relation (1.22).

From (1.21) we obtain

$$d\dot{x}^\gamma = F(\delta^\gamma_\theta - A_{\circ\ \theta}^{\ \gamma}) Dl^\theta - F\Gamma^{*\gamma}_{o\ \delta} dx^\delta - \dot{x}^\gamma F dF^{-1} \tag{2.8}$$

Substituting (2.8) into (2.2) we have

$$\Gamma^\alpha_{\beta\gamma} - F C^\alpha_{\beta\delta} \Gamma^{*\delta}_{o\ \gamma} = \Gamma^{*\alpha}_{\beta\ \gamma} + \theta^\alpha_\beta \lambda_\gamma \tag{2.9}$$

$$F C^\alpha_{\beta\delta} (\delta^\delta_\gamma - A_{\circ\ \gamma}^{\ \alpha}) = A^\alpha_{\beta\gamma} + \theta^\alpha_\beta (\mu_\gamma + 2\, l_\gamma) \tag{2.10}$$

In the case of non recurrent Finsler space where $\lambda_\gamma = 0$, $\mu_\gamma = 0$, $A_{\circ\ \gamma}^{\ \delta} = 0$ the equations (2.9) and (2.10) have the from

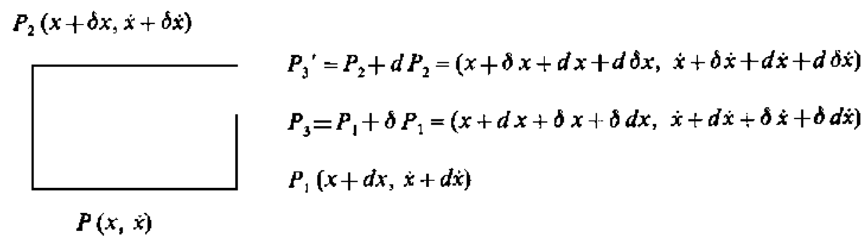
$$\Gamma^\alpha_{\beta\gamma} - F C^\alpha_{\beta\delta} \Gamma^{*\delta}_{o\ \gamma} = \Gamma^{*\alpha}_{\beta\ \gamma} \tag{2.11}$$

$$FC^{\alpha}_{\beta\gamma} = A^{\alpha}_{\beta\gamma} + 2\theta^{\alpha}_{\beta} l_{\gamma} \tag{2.12}$$

For $\theta^{\alpha}_{\beta} = 0$ (2.12) takes the well known form $FC^{\alpha}_{\beta\gamma} = A^{\alpha}_{\beta\gamma}$. In the further calculation we shall use the formulae [4]

$$F_{|\gamma} = \partial_{\gamma} F - F\dot{\partial}_{\delta} F \Gamma^{*\delta}_{\circ\ \gamma} = 2^{-1} F \lambda_{\gamma}.$$

3. Parallel displacenxent of vector along the cycle of lineelements. Let us consider the cycle of lineelements as they are presented on the picture



Let us fix the point P with the local coordinates x^{α} in $\overline{F}_n$. By $T_n(P)$ we shall denote the set of all $\dot{x}$ in P which form a tangent space. In $T_n(P)$ we can construct a basis which containes the tangent vectors r_{α} $(\alpha = 1, 2, \dots, n)$ on the coordinate curves $x^{\beta} = C^{\beta}$, $\beta = 1, 2, \dots, \alpha-1, \alpha+1, \dots, n$. Let us consider two infinitesimal vectors PP_1 and PP_2 which respectivly have the form $PP_1 = dx^{\alpha} r_{\alpha}$, $PP_2 = \delta x^{\alpha} r_{\alpha}$. If the vector PP_1 is parallely transported along PP_2 we get the point P_3 and if PP_2 is parallely moved along PP_1 we get P_3'. In this case the lineelement are not parallel, only the basic vectors are. The coordinates of the point P_3 are $x^{\alpha} + dx^{\alpha} + \delta x^{a} + \delta dx^{a}$, where $\delta dx^{a} = -w^{\alpha}_{\beta}(\delta) dx^{\beta}$ and the coordinates of the point P_3' are $x^{\alpha} + \delta x^{\alpha} + dx^{a} + d\delta x^{a}$ where $d\delta x^{a} = -w^{\alpha}_{\beta}(d)\delta x^{\beta}$. In the general case P_3 and P_3' are not the same points and the vector P_3P_3' is the torsion vector in $\overline{F}_n$. It has the coordinates

$$\Omega^{\alpha} = d\delta x^{a} - \delta dx^{a} = w^{\alpha}_{\beta}(\delta) dx^{\beta} - w^{\alpha}_{\beta}(d)\delta x^{\beta}$$

In $\overline{F}_n$ with the connection coefficients Γ^* and A we obtain

$$\Omega^{\alpha} = A^{\alpha}_{\beta\gamma}(dx^{\beta} \Delta l^{\gamma} - \delta x^{\beta} D l^{\gamma}).$$

If $Dl^{\gamma} = 0$ and $\Delta l^{\gamma} = 0$, then $\Omega^{\alpha} = 0$ and the points P_3 and P_3' have the same coordinates. In that case we have an infinitesimal parallelogram $PP_1P_2P_3$.

Let us consider how the basic vectors change if they are parallely transported along PP_1P_3 and $PP_2P_3'P_3$.

By the parallel transportation of r_{α} from $P(x, \dot{x})$ to $P_1(x + dx, \dot{x} + d\dot{x})$ we obtain in $P_1 r_{\alpha} + dr_{\alpha}$, where $Dr_{\alpha} = dr_{\alpha} - w_{\alpha}{}^{\beta}(d) r_{\beta} = 0$.

By the parallel transportation of r_α from $P(x,\dot{x})$ to $P_2(x+\delta x, \dot{x}+\delta\dot{x})$ in P_2 we get $r_\alpha + \delta r_\alpha$, where $\Delta r_\alpha = \delta r_\alpha - w_\alpha{}^\beta(d) r_\beta = 0$.

If the vector $r_\alpha + dr_\alpha$ at P_1 is parallely transported to $P_3(x+dx+\delta x + \delta dx, \dot{x}+d\dot{x}+\delta\dot{x}+\delta d\dot{x})$ at P_3 we have the vector $r_\alpha + dr_\alpha + \delta(r_\alpha + dr_\alpha)$, where

$$\delta dr_\alpha = \delta w_\alpha{}^\beta(d) r_\beta + w_\alpha{}^\delta(d) w_\delta{}^\beta(\delta) r_\beta.$$

If the vector $r_\alpha + \delta r_\alpha$ at P_2 is parallely transported to $P_3'(x+\delta x+dx + d\delta x, \dot{x}+\delta\dot{x}+d\dot{x}+d\delta\dot{x})$ at P_3' we get the vector $r_\alpha + \delta r_\alpha + d(r_\alpha + \delta r_\alpha)$ where

$$d\delta r_\alpha = dw_\alpha{}^\beta(\delta) r_\beta + w_\alpha{}^\beta(\delta) w_\delta{}^\beta(d) r_\beta.$$

If the vector $r_\alpha + \delta r_\alpha + dr_\alpha + d\delta r_\alpha$ at P_3' is parallely transported to P_3 we obtain in P_3 the vector $r_\alpha + \delta r_\alpha + dr_\alpha + d\delta r_\alpha + \nabla r_\alpha$ where ∇r_α describes the change of r_α along $P_3' P_3$ and has the form

$$\nabla r_\alpha = \Gamma_\alpha{}^\beta{}_\gamma r_\beta(\delta d - d\delta) x^\gamma + C_\alpha{}^\beta{}_\gamma r_\beta(\delta d - d\delta)\dot{x}^\gamma.$$

The difference between vectors which are obtained by parallel transportation of r_α along $PP_2P_3'P_3$ and PP_2P_3 is denoted by $\overline{D} r_\alpha$. Then we have

$$\begin{aligned} \overline{D} r_\alpha = &- (\delta d - d\delta) r_\alpha + \nabla r_\alpha = \\ &- (\delta d - d\delta) r_\alpha + \Gamma_\alpha{}^\beta{}_\gamma r_\beta(\delta d - d\delta) x^\gamma + C_\alpha{}^\beta{}_\gamma r_\beta(\delta d - d\delta)\dot{x}^\gamma. \end{aligned} \tag{3.1}$$

The vector $\overline{D} r_\alpha$ can be expressed by the curvature tensors. We have $Dr_\alpha = dr_\alpha - w_\alpha{}^\beta(d) r_\beta$ and

$$\begin{aligned} \Delta\, Dr_\alpha = \delta\,(Dr_a) - w_\alpha{}^\delta(\delta)\, Dr_\delta = \delta\, dr_\alpha - \delta\, w_\alpha{}^\beta(d) r_\beta - \\ w_\alpha{}^\beta(d)\,\delta\, r_\beta - w_\alpha{}^\delta(\delta)\,[dr_\delta - w_\delta{}^\beta(d)\, r_\beta]. \end{aligned}$$

From the above equation we get

$$(\Delta D - D\Delta) r_\alpha = (\delta d - d\delta) r_\alpha - \Omega_\alpha{}^\beta r_\beta, \tag{3.2}$$

where

$$\begin{gathered} w_\alpha{}^\beta = [w_\alpha{}^\delta w_\delta{}^\beta] - (w_\alpha{}^\beta)' \\ [w_\alpha{}^\delta w_\delta{}^\beta] = w_\alpha{}^\delta(d) w_\delta{}^\beta(\delta) - w_\alpha{}^\delta(\delta) w_\delta{}^\beta(d) \\ (w_\alpha{}^\beta)' = \delta w_\alpha{}^\beta(d) - dw_\alpha{}^\beta(\delta). \end{gathered}$$

After some calculation we obtain

$$\Omega_\alpha{}^\beta = A_\alpha{}^\beta + B_\alpha{}^\beta, \tag{3.3}$$

where [5]

$$A_\alpha{}^\beta = 2^{-1} K_\alpha{}^\beta{}_{\gamma\delta}[dx^\beta \delta x^\gamma] + (P'_\alpha{}^\beta{}_{\gamma\delta} - A_\alpha{}^\beta{}_\iota \dot{\partial}_\delta \Gamma^{*\iota}_\delta) + 2^{-1} S_\alpha{}^\beta{}_{\gamma\delta}[Dl^\gamma \Delta l^\delta] \tag{3.4}$$

$$B_\alpha{}^\beta = A_\alpha{}^\beta{}_\gamma(\delta D - d\Delta)l^\gamma + \Gamma^{*\beta}_{\alpha\gamma}(\delta d - d\delta)x^\gamma \tag{3.5}$$
$$2^{-1}K_\alpha{}^\beta{}_{\gamma\delta} = \partial_{[\delta}\Gamma^{*\beta}_{|\alpha|\gamma} - \dot\partial_\iota\Gamma^{*\beta}_{\alpha[\gamma}\Gamma^{*\iota}_{\delta]} + \Gamma^{*\iota}_{\alpha[\gamma}\Gamma^{*\beta}_{|\iota|\delta]}.$$

$$P'_\alpha{}^\beta{}_{\gamma\delta} = F\dot\partial_\iota\Gamma^{*}_\alpha{}^\beta{}_\gamma(\delta^\iota_\delta - A_o{}^\iota_\delta) - A^\beta_{\alpha\delta|\gamma} + A^\beta_{\alpha\iota}\dot x^\chi\dot\partial_\delta\Gamma^{*\iota}_{\chi\gamma} \tag{3.6}$$
$$2^{-1}S_\alpha{}^\beta{}_{\gamma\delta} = F\dot\partial_\iota A^{*}_\alpha{}^\beta{}_{[\gamma}(\delta^\iota_{\delta]} - A^\iota_{|o|\delta]}) + A_\alpha{}^\iota{}_{[\gamma}A_{|\iota|}{}^\beta{}_{\delta]}.$$

On the other hand from (1.4) and (2.8) using the homogenity of $\Gamma^{*\iota}_\gamma = F\,\Gamma^{*\iota}_{o\gamma}$ (first degree) and $A^\iota_{o\gamma}$ (zero degree) we obtain

$$(\delta^\chi_\delta - A_o{}^\chi_\delta)(\delta D - d\Delta)l^\delta = B^\chi + \overline{B}^\chi \tag{3.7}$$

where

$$\begin{aligned}\overline{B}^\chi =& F^{-1}(\partial_{[\delta}\Gamma^{*\chi}_{\gamma]} - \dot\partial_\iota\Gamma^{*\chi}_{[\gamma}\Gamma^{*i}_{\delta]}[dx^\gamma\delta x^\delta] + \\ &(\dot\partial_\delta\Gamma^{*\chi}_\gamma - \dot\partial_\iota\Gamma^{*}{}^\chi_\gamma A_o{}^\iota{}_\delta - \partial_\gamma A_o{}^\iota{}_\delta + \dot\partial_\iota A_o{}^\chi{}_\delta\Gamma^{*}_\gamma{}^\iota)[dx^\gamma\,\Delta l^\delta] + \\ &F\dot\partial_{[\delta}A_{|o|}{}^\chi{}_{\gamma]} - \dot\partial_\iota\Gamma^{*}_o{}^\chi{}_{[\gamma}A_{|o|}{}^\iota{}_{\delta]} + [Dl^\gamma\Delta l^\delta]\end{aligned}$$

$$B^\chi = F^{-1}(\delta d - d\delta)\dot x^\chi + \dot x^\chi(\delta d - d\delta)F^{-1} + F^{-1}\Gamma^{*}_\gamma{}^\chi(\delta d - d\delta)x^\gamma \tag{3.8}$$

It is known that $\dot x^\alpha{}_{|\beta}$, so from the above equation and (3.4) we obtain

$$2^{-1}K_o{}^\chi{}_{\gamma\delta} = F^{-1}(\partial_{[\delta}\Gamma^{*\chi}_{\gamma]} - \dot\partial_\iota\Gamma^{*\chi}_{[\gamma}\Gamma^{*\chi}_{\delta]}) \tag{3.9}$$

Substituting (2.10) into (3.5) we get

$$B_\alpha{}^\beta = B_\alpha{}^\beta{}_{(1)} + B_\alpha{}^\beta{}_{2)} \tag{3.10}$$

where according to (3.7) we have

$$B_\alpha{}^\beta{}_{(1)} = \Gamma^{*\beta}_{\alpha\gamma}(\delta d - d\delta)x^\gamma + FC_\alpha{}^\beta{}_\chi B^\chi - \theta_\alpha{}^\beta(\mu\delta + 2l_\delta)(\delta D - d\Delta)\,l^\delta,$$

$$B_\alpha{}^\beta{}_{(2)} = FC_\alpha{}^\beta{}_\chi\overline{B}^\chi \tag{3.11}$$

From (1.18) we get

$$\begin{aligned}&\lambda_\gamma(\delta d - d\delta)\,x^\gamma + (\mu_\gamma + 2l_\gamma)(\delta D - d\Delta)\,l^\gamma + (\delta\lambda_\gamma\,dx^\gamma - d\lambda_\gamma\,\delta x^\gamma) + \\ &\delta(\mu_\gamma + 2l_\gamma)\,Dl^\gamma - d\,(\mu_\gamma + 2l_\gamma)\,\Delta l^\gamma = 0\end{aligned} \tag{3.12}$$

and using (3.18) and (2.5) we have

$$\begin{aligned}B_\alpha{}^\beta{}_{(1)} =& (\Gamma^{*\beta}_{\alpha\gamma} + C_\alpha{}^\beta{}_\chi\Gamma^{*\chi}_\gamma + \theta_\alpha{}^\beta\lambda_\gamma)(\delta d - d\delta)\,x^\gamma \\ &C_\alpha{}^\beta{}_\gamma(\delta d - d\delta)\,\dot x^\gamma + B_\alpha{}^\beta{}_{(1)'}\end{aligned} \tag{3.13}$$

$$\begin{aligned}B_\alpha{}^\beta{}_{(1)'} =& \theta_\alpha{}^\beta[\delta\lambda_\gamma dx^\gamma - d\lambda_\gamma\,\delta x^\gamma \\ &\delta(\mu_\gamma + 2l_\gamma)\,Dl^\gamma - d\,(\mu_\gamma + 2l_\gamma)\,\Delta l^\gamma].\end{aligned} \tag{3.14}$$

Substituting $\Gamma_\alpha{}^\beta{}_\gamma$ from (2.9) into (3.13) we have

$$B_\alpha{}^\beta{}_{(1)} = \Gamma^{*}_\alpha{}^\beta{}_\gamma(\delta d - d\delta)\, x^\gamma + C_\alpha{}^\beta{}_\gamma(\delta d - d\delta)\, \dot{x}^\gamma + B_\alpha{}^\beta{}_{(1)'}$$

Using (3.9) and the relation

$$\dot{\partial}_\iota \Gamma^{*\chi}_\gamma(\delta^\iota_\delta - A_o{}^\iota{}_\delta) - \partial_\gamma A_o{}^\chi{}_\delta + \dot{\partial}_\iota A^\chi_{o\delta}\Gamma^{*\iota}_\gamma = P'^{\chi}_o{}_{\gamma\delta} - A_o{}^\chi{}_\iota\dot{\partial}_\delta\Gamma^{*\iota}_\gamma + \Gamma^{*\chi}_{\delta\gamma} + 2^{-1}A_o{}^\chi_\delta\lambda_\gamma \tag{3.15}$$

the formula (3.11) has the form

$$\begin{aligned} B_\alpha{}^\beta{}_{(2)} =& FC_\alpha{}^\beta{}_\chi\{2^{-1}K_o{}^\chi{}_{\gamma\delta}[dx^\gamma\, \varDelta l^\delta]+ \\ & (P'^{\chi}_o{}_{\gamma\delta} - A_o{}^\chi{}_\iota\dot{\partial}_\delta\Gamma^{*\iota}_\gamma + \Gamma^{*\chi}_{\delta\gamma} + 2^{-1}A_o{}^\chi{}_\delta\lambda_\gamma)[dx^\gamma\, \varDelta l^\delta]+ \\ & 2^{-1}(\dot{\partial}_\iota A_o{}^\chi{}_{[\gamma}(\partial^\iota_{\delta]} - A_{|o|}{}^\iota{}_{\delta]})[Dx^\gamma\, \varDelta l^\delta]\}. \end{aligned} \tag{3.16}$$

Theorem 3.1. *In the recurrent Finsler space $F_n\overline{D}r_\alpha$ and the curvature tensors are connected by:*

$$\begin{aligned} (\varDelta D - D\varDelta)\, r_\alpha =& -\overline{D}r_\alpha - r_\beta\{2^{-1}[K_\alpha{}^\beta{}_{\gamma\delta} + FC_\alpha{}^\beta{}_\chi K_o{}^\chi{}_{\gamma\delta}][dx^\gamma\, \delta x^\delta]+ \\ & [P'^{\beta}_\alpha{}_\gamma\delta - A_\alpha{}^\beta{}_\iota\dot{\partial}_\delta\Gamma^{*\iota}_\gamma + FC_\alpha{}^\beta{}_\chi(P'^{\chi}_o{}_{\gamma\delta} - A_o{}^\chi{}_\iota\dot{\partial}_\delta\Gamma^{*\iota}_\gamma + \Gamma^{*\chi}_{\delta\gamma} + 2^{-1}A_o{}^\chi{}_\delta\lambda_\gamma)[dx^\gamma\, \varDelta l^\delta]+ \\ & 2^{-1}[S_\alpha{}^\beta{}_{\gamma\delta} + F^2C_\alpha{}^\beta{}_\chi\dot{\partial}_\iota A_o{}^\chi{}_{[\gamma}(\delta^\iota_{\delta]} - A_{|o|}{}^\iota{}_{\delta]})]][Dx^\gamma\, \varDelta l^\delta] - r_\beta B_\alpha{}^\beta{}_{(1)'} \end{aligned} \tag{3.17}$$

Proof. Substituting (3.16), (3.13), (3.14) into (3.10), further (3.10) and (3.4) into (3.3), (3.4) into (3.2) by using (3.1) we obtain (3.17).

In the non recurrent Finsler space (where $\lambda_\gamma = 0$ and $\mu_\gamma = 0$ we have

$$B_\alpha{}^{\beta\prime}{}_{(1)} = 2\theta_\alpha{}^\beta(\delta l_\gamma Dl^\gamma - dl_\gamma\Delta l^\gamma).$$

If we have not only $\lambda_\gamma = 0$, $\mu_\gamma = 0$ but the condition $\theta_\alpha{}^\beta = 0$, then the connection coefficients $A_\alpha{}^\beta{}_\gamma$ and $\Gamma^{*}_\alpha{}^\beta{}_\gamma$ are the Cartans connection coefficients and $A_\alpha{}^\beta{}_\gamma = FC_\alpha{}^\beta{}_\gamma$. In this case from (1.11), (1.12) it follows $A_o{}^\chi{}_\gamma = 0$ the left hand side of (3.15) reduces to the $\dot{\partial}_\delta\Gamma^{*\chi}_\gamma$ and (3.17) has the form

$$\begin{aligned} &(\Delta D - D\Delta)r_\alpha = \\ &-\overline{D}r_\alpha - r_\beta\{2^{-1}R_\alpha{}^\beta{}_{\gamma\delta}[dx^\gamma\delta x^\delta] + P'^{\beta}_\alpha{}_{\gamma\delta}[dx^\gamma\varDelta l^\delta] + 2^{-1}S_\alpha{}^\beta{}_{\gamma\delta}[Dl^\gamma\Delta l^\delta]\}. \end{aligned} \tag{3.18}$$

When the vector r_a is parallely transported along PP_1P_3 and $PP_2P'_3P_3$ then $Dr_\alpha = 0$, $\varDelta r_\alpha = 0$ and in this case from (3.18) we have

$$-\overline{D}r_\alpha = -r_\beta\{2^{-1}R_\alpha{}^\beta{}_{\gamma\delta}[dx^\gamma\delta x^\delta] + P'^{\beta}_\alpha{}_{\gamma\delta}[dx^\gamma\varDelta l^\delta] + 2^{-1}S_\alpha{}^\beta{}_{\gamma\delta}[Dl^\gamma\Delta l^\delta]\}.$$

In the case of a recurrent Finsler space $\overline{F}_n$ when $Dr_\alpha = 0$ and $\varDelta r_\alpha = 0$ from (3.17) $\overline{D}r_\alpha$ has more complicated form.

4. Special cases- *Case* 1. Let us consider the case when in $\overline{F}_n$, $dx^\gamma = 0$ and $\delta x^\gamma = 0$ i. e. when the lineelements P, P_1 and P_2 have the common center x.

Then we have

$$\begin{aligned} &P(x,\dot{x}),\ P_1(x,\dot{x}+d\dot{x}),\ P_2(x,\dot{x}+\delta\dot{x}) \\ &P_3 = P_1 + \delta P_1 = (x, \dot{x} + d\dot{x} + \delta\dot{x} + \delta d\dot{x}), \\ &P_3' = P_2 + dP_2 = (x, \dot{x} + \delta\dot{x} + d\dot{x} + d\delta\dot{x}). \end{aligned}$$

In this case we have

$$Dr_\alpha = dr_\alpha - A_\alpha{}^\beta{}_\gamma r_\beta Dl^\gamma, \quad \Delta r_\alpha = \delta r_\alpha - A_\alpha{}^\beta{}_\gamma r_\beta \Delta l^\gamma$$

and

$$\begin{aligned} (\Delta - D\Delta)\, r_\alpha =& (\delta d - d\delta)\, r_a - 2^{-1} r_\beta [F\dot{\partial}_\iota A_\alpha{}^\beta{}_{[\gamma}\delta^\iota_{\delta]} - A_{|o|}{}^\iota{}_{\delta]}) + \\ & A_\alpha{}^\iota{}_{[\delta} A_{|\iota|}{}^\beta{}_{\gamma]}][Dl^\gamma \Delta l^\delta] - A_\alpha{}^\beta{}_\gamma r_\beta (\delta D - d\Delta)\, l^\gamma. \end{aligned} \tag{4.1}$$

Substituting $A_\alpha{}^\beta{}_\gamma$ from (2.10) and using (3.12) where $(\delta d - d\delta)\, x^\gamma = 0$ we have

$$\begin{aligned} -A_\alpha{}^\beta{}_\chi\, r_\beta\, (\delta D - d\Delta)\, l^\chi =& - FC_\alpha{}^\beta{}_\iota\, r_\beta\, (\delta^\iota_\chi - A_o{}^\iota{}_\chi)\, (\delta D - d\Delta)\, l^\chi \\ & - \theta_\alpha{}^\beta r_\beta [\delta(\mu_\gamma + 2l_\gamma) Dl^\gamma - d(\mu_\gamma + 2l_\gamma)\Delta l^\gamma]. \end{aligned} \tag{4.2}$$

As in this case

$$(\delta^\iota_\chi - A_o{}^\iota{}_\chi) Dl^\chi = dl^\iota, \quad (\delta^\iota_\chi - A_o{}^\iota{}_\chi)\Delta l^\chi = \delta l^\iota$$

using the homogenity condition we obtain

$$\begin{aligned} (\delta^\iota_\chi - A_o{}^\iota{}_\chi)(\delta D - d\Delta) l^\chi =& F^{-1}(\delta d - d\delta)\dot{x}^\iota + \dot{x}^\iota(\delta d - d\delta)F^{-1} + \\ & F\dot{\partial}_\chi A_o{}^\iota{}_\gamma (\delta^\chi_\delta - A_o{}^\chi_\delta)(Dl^\gamma\, \Delta l^\delta - \Delta l^\gamma \Delta l^\delta]. \end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.2) and then (4.2) into (4.1) we get

$$\begin{aligned} (\Delta D - D\Delta) r_\alpha = -\overline{D} r_\alpha - r_\beta [2^{-1} S_\alpha{}^\beta{}_{\gamma\delta} + F^2 C_\alpha{}^\beta{}_\chi \dot{\partial}_\iota A_o{}^\chi{}_{[\gamma} (\partial^\iota_{\delta]} - A_{|o|}{}^\iota{}_{\delta]})][Dl^\gamma\, \Delta l^\delta] - \\ \theta_\alpha{}^\beta r_\beta [\delta(\mu_\gamma + 2l_\gamma)\, Dl^\gamma - d\,(\mu_\gamma + 2l_\gamma)\, \Delta l^\gamma] \end{aligned}$$

where from (3.1) in this case $\overline{D} r_\alpha$ has the form

$$\overline{D} r_\alpha = -(\delta d - d\delta) r_\alpha + C_\alpha{}^\beta{}_\gamma r_\beta (\delta d - d\delta)\dot{x}^\gamma$$

In the non-recurrent Finsler space F_n, where we take $\theta_\alpha{}^\beta = 0$, $\mu_\gamma = 0 \Rightarrow A_\alpha{}^\beta{}_\gamma = FC_\alpha{}^\beta{}_\gamma \Rightarrow A_o{}^\beta{}_\gamma = 0$ we have

$$(\Delta D - D\Delta) r_\alpha = -\overline{D} r_\alpha - 2^{-1} r_\beta S_{\alpha\beta\gamma\delta} [Dl^\gamma\, \Delta l^\delta]. \tag{4.4}$$

In the case when $Dr_\alpha = 0$, $\Delta r_a = 0$ (4.4) gives

$$\overline{D} r_\alpha = -2^{-1} r_\beta S_\alpha{}^\beta{}_{\gamma\delta} [Dl^\gamma\, \Delta l^\delta]$$

Case 2. Let us consider the lineelements

$P(x, \dot{x})$

$P_1(x + dx, \dot{x} + \delta\dot{x}) \quad \text{with} \quad Dl = 0$

$P_2(x, \dot{x} + \delta\dot{x}) \quad \text{with} \quad \delta x = 0$

$P_3 = P_1 + \delta P_1 = (x + dx, \dot{x} + d\dot{x} + \delta\dot{x} + \delta d\dot{x}),\ (\delta x = 0),$

$P'_3 = P_2 + dP_2 = (x + dx, \dot{x} + \delta\dot{x} + d\dot{x} + d\delta\dot{x}).$

From $Dl^\delta = 0$ we have

$$d\dot{x}^\delta = -F\dot{x}^\delta dF^{-1} - \mathit{\Gamma}^{*\delta}_{\ \gamma} dx^\gamma. \tag{4.5}$$

From $\delta x^\delta = 0$ we get

$$\begin{aligned}(\delta^\delta_\iota - A_o{}^\delta{}_\iota)\,\mathit{\Delta} l^\iota = \delta l^\delta = F^{-1}\delta\dot{x}^\delta + \dot{x}^\delta\delta F^{-1} \Rightarrow \\ \delta\dot{x}^\delta = (\delta^\delta_\iota - A_o{}^\delta{}_\iota)\,\mathit{\Delta} l^\iota - F\dot{x}^\delta\delta F^{-1}.\end{aligned} \tag{4.6}$$

In this case we have

$$\text{a) } Dr_\alpha = dr_\alpha - \mathit{\Gamma}^{*\beta}_{\alpha\gamma} r_\beta \delta\dot{x}^\gamma \quad \text{b) } \mathit{\Delta} r_\alpha = \delta r_\alpha - A_\alpha{}^\beta{}_\gamma r_\beta \mathit{\Delta} l^\gamma \tag{4.7}$$

From $\delta x = 0 \Rightarrow d\delta x = 0$ and $\overline{D} r_\alpha$ has the form

$$-\overline{D} r_\alpha = -(\delta d - d\delta) r_\alpha + \mathit{\Gamma}_\alpha{}^\beta{}_\gamma r_\beta \delta x^\gamma - C_\alpha{}^\beta{}_\gamma r_\beta (\delta d - d\delta)\dot{x}^\gamma \tag{4.8}$$

From (4.7) we obtain

$$\begin{aligned}(\mathit{\Delta} D - D\mathit{\Delta})\, r_\alpha = r_\beta\, [F\dot{\partial}_\iota \Gamma^{*}_{\alpha}{}^{\beta}{}_{\gamma} (\partial^\iota_\delta - A_o{}^\iota{}_\delta) - \partial_\iota A_\alpha{}^\beta{}_\delta + \dot{\partial}_\iota A_\alpha{}^\beta{}_\delta \Gamma^{*}_{\gamma}{}^{\iota} \\ -A_\alpha{}^\iota{}_\delta \Gamma^{*}_{\iota}{}^{\beta}{}_{\gamma} + A_\iota{}^\beta{}_\delta \Gamma^{*}_{\alpha}{}^{\iota}{}_{\gamma}]\, dx^\gamma\, \mathit{\Delta} l^\delta + \\ (\delta d - d\delta) r_\alpha - \Gamma^{*}_{\alpha}{}^{\beta}{}_{\gamma} r_\beta \delta dx^\gamma + A_\alpha{}^\beta{}_\gamma r_\beta d\mathit{\Delta} l^\gamma.\end{aligned} \tag{4.9}$$

From (2.10) using (4.6) and $C_\alpha{}^\beta{}_\chi \dot{x}^\chi = 0$ we get

$$A_\alpha{}^\beta{}_\gamma\, r_\beta\, d\mathit{\Delta} l^\gamma = [FC_\alpha{}^\beta{}_\iota\, r_\beta\, (\delta^\iota_\gamma - A_o{}^\iota{}_\gamma) - \theta_\alpha{}^\beta\, (\mu_\gamma + 2l_\gamma)]d\,\mathit{\Delta} l^\gamma.$$

From (3.12) in case 2 it follows

$$B = (\mu_\gamma + 2l_\gamma) d\,\mathit{\Delta} l^\gamma = \lambda_\gamma \delta\, dx^\gamma + \delta\lambda_\gamma dx^\gamma - d(\mu_\gamma + 2l_\gamma)\,\mathit{\Delta} l^\gamma.$$

From Lemma 1.3 it follows that in case

$$Dl^\gamma = 0 \Rightarrow \lambda_\gamma dx^\gamma = 0 \Rightarrow \delta\lambda_\gamma dx^\gamma + \lambda_\gamma \delta\, dx^\gamma = 0$$

and B reduces to the form $B = -d(\mu_\gamma + 2l_\gamma)\,\mathit{\Delta} l^\gamma$. Then

$$\begin{aligned}A_\alpha{}^\beta{}_\gamma\, d\mathit{\Delta} l^\gamma = FC_\alpha{}^\beta{}_\chi\, r_\beta\, (\partial_\gamma A_o{}^\chi{}_\delta - \dot{\partial}_\iota A_o{}^\chi{}_\delta \Gamma^{*}_{\gamma}{}^{\iota})\, dx^\gamma \mathit{\Delta} l^\delta - \\ \theta_\alpha{}^\beta\, r_\beta B + FC_\alpha{}^\beta{}_\delta\, r_\beta\, (dF^{-1}\delta\dot{x}^\delta + F^{-1} d\delta\dot{x}^\delta + d\dot{x}^\delta \delta F^{-1}).\end{aligned} \tag{4.10}$$

We can add and substract $\delta d\dot{x}^\delta$, to the last term of (4.10), where from (4.5) we have

$$\delta d\dot{x}^\delta = -\delta F \dot{x}^\delta dF^{-1} - F\delta\dot{x}^\delta dF^{-1} - F\dot{x}^\delta \delta dF^{-1} - \dot{\partial}_\iota \Gamma^{*\delta}_{\chi}[F(\partial^\iota_\gamma - A_o{}^\iota{}_\gamma)\,\Delta l^\gamma - F\dot{x}^\iota \delta F^{-1}]dx^\chi - \Gamma^{*\delta}_{\chi}\delta dx^\chi.$$

Using the homogenity condition of $\Gamma^{*\delta}_{\chi}$ in $\dot{x}$ (first degree) and the relation $C_\alpha{}^\beta{}_\delta \dot{x}^\delta = 0$ (4.10) has the form

$$A_\alpha{}^\beta{}_\gamma\, r_\beta\, d\Delta l^\gamma = -FC_\alpha{}^\beta{}_\chi\, r_\beta\,[\dot{\partial}_i \Gamma^{*\chi}_{\gamma}(\partial^\iota_\delta - A_o{}^\iota{}_\delta) - \partial_\gamma\, A_o{}^\chi{}_\delta + \dot{\partial}_\iota\, A_o{}^\chi{}_\delta \Gamma^{*\iota}_{\gamma}]\, dx^\gamma\, \Delta l^\delta - C_\alpha{}^\beta{}_\delta\, r_\beta\,(\delta d - d\delta)\,\dot{x}^\delta - C_\alpha{}^\beta{}_\chi\, r_\beta\, \Gamma^{*\chi}_{\gamma}\, \delta\, dx^\gamma - \theta_\alpha{}^\beta\, r_\beta\, B. \tag{4.11}$$

Substituting (4.11) into (4.9) using (3.6), (3.15), (4.8) and (2.9) we obtain

$$(\Delta D - D\Delta)\, r_\alpha = -\overline{D} r_\alpha - r_\beta\,[P'^{\,\beta}_{\alpha}{}_{\gamma\delta} - A_\alpha{}^\beta{}_\iota \dot{\partial}_\delta \Gamma^{*\iota}_{\gamma} + FC_\alpha{}^\beta{}_\chi (P'^{\,\chi}_{o}{}_{\gamma\delta} - A_o{}^\chi{}_\iota \dot{\partial}_\delta \Gamma^{*\iota}_{\gamma} + \Gamma^{*\chi}_{\delta}{}_\gamma + 2^{-1} A_o{}^\chi{}_\delta \lambda_\gamma]\, dx^\gamma\, \Delta l^\delta - \theta_\alpha{}^\beta r_\beta\, B. \tag{4.12}$$

In the non recurrent Finsler space F_n when $\theta_\alpha{}^\beta = 0$ (4.12) reduces to the form

$$(\Delta D - D\Delta)\, r_\alpha = -\overline{D} r_\alpha - r_\beta\, P'^{\,\beta}_{\alpha}{}_\gamma\, \delta\, dx^\gamma\, \Delta l^\delta.$$

When $Dr_\alpha = 0$, $\Delta\, r_\alpha = 0$ from (4.13) it is easy to see that

$$\overline{D} r_\alpha = -r_\beta\, P'^{\,\beta}_{\alpha}{}_{\gamma\delta}\, dx^\gamma\, \Delta l^\delta.$$

Case 3. Let us consider the cycle of lintlements

$$P(x, \dot{x}),$$

$$P_1(x + dx, \dot{x} + d\dot{x}), \quad Dl^\alpha = 0 \Rightarrow d\dot{x}^\alpha = \dot{x}^\alpha F^{-1} dF - \Gamma^{*\alpha}_{\beta} dx^\beta, \tag{4.14}$$

$$P_2(x + \delta x, \dot{x} + \delta\dot{x}), \quad \Delta l^\alpha = 0 \Rightarrow \delta\dot{x}^\alpha = \dot{x}^\alpha F^{-1}\delta F - \Gamma^{*\alpha}_{\beta}\delta x^\beta, \tag{4.15}$$

$$P_3 = P_1 + \delta P_1 = (x + dx + \delta x + \delta dx, \dot{x} + d\dot{x} + \delta\dot{x} + \delta d\dot{x}),$$

$$P'_3 = P_2 + dP_2 = (x + \delta x + dx + d\delta x, \dot{x} + \delta\dot{x} + d\dot{x} + d\delta\dot{x}).$$

From $Dr_\alpha = dr_\alpha - \Gamma^{*\beta}_{\alpha}{}_\gamma r_\beta dx^\gamma$ it follows

$$(\Delta D - D\Delta)\, r_\alpha = (\delta d - d\delta)\, r_\alpha - \Gamma^{*\beta}_{\alpha}{}_\gamma r_\gamma (\delta d - d\delta) x^\gamma - 2^{-1} K_\alpha{}^\beta{}_{\gamma\delta}[dx^\gamma\, \delta x^\delta]. \tag{4.16}$$

From (4.14), (4.15) and $C_\alpha{}^\beta{}_\gamma \dot{x}^\gamma = 0$ it follows

$$C_\alpha{}^\beta{}_\gamma (\delta d - d\delta)\dot{x}^\gamma = C_\alpha{}^\beta{}_\theta \Gamma^{*\theta}_{\beta}(\delta d - d\delta) x^\beta - 2^{-1} FC_\alpha{}^\beta{}_\theta K_o{}^\theta{}_{\beta\gamma}[dx^\beta\, \delta x^\gamma]. \tag{4.17}$$

From (4.17) and (2.9) we obtain

$$\Gamma^{*\beta}_{\alpha}{}_\gamma (\delta d - d\delta)\, x^\gamma = (\Gamma_\alpha{}^\beta{}_\gamma - \theta_\beta{}^\alpha \lambda_\gamma)\,(\delta d - d\delta)\, x^\gamma + C^{*\beta}_{\alpha}{}_\gamma (\delta d - d\delta)\,\dot{x}^\gamma + 2^{-1} FC_\alpha{}^\beta{}_\theta K_o{}^\theta{}_{\beta\gamma}[dx^\beta\, \delta x^\gamma]. \tag{4.18}$$

Substituting (4.18) into (4.16) and using (3.1) we get

$$(\varDelta D - D\varDelta)\, r_\alpha = -\overline{D} r_\alpha - 2^{-1}(K_\alpha{}^\beta{}_{\gamma\delta} + FC_\alpha{}^\beta{}_\chi K^o{}_{\chi\gamma\delta})\,[dx^\gamma\, \delta x^\delta] + \theta_\alpha{}^\beta\, \lambda_\gamma\, (\delta d - d\delta)\, x^\gamma.$$

For the case of a non recurrent Finsler space (when $\lambda_\gamma = 0$, $\mu_\gamma = 0$) and $\theta_\alpha{}^\beta = 0$ $\varGamma_\alpha{}^\beta{}_\gamma$ and $A_\alpha{}^\beta{}_\gamma = FC_\alpha{}^\beta{}_\gamma$ are the Cartans connection coefficients. In this case for $Dr_\alpha = 0$ and $\varDelta r_\alpha = 0$ we obtain.

$$\overline{D} r_\alpha = -2^{-1} R_\alpha{}^\beta{}_{\gamma\delta} r_\beta\, [dx^\gamma\, \delta x^\delta]$$

where $R_\alpha{}^\beta{}_{\gamma\delta} = K_\alpha{}^\beta{}_{\gamma\delta} + A_\alpha{}^\beta{}_\chi K_\chi{}^o{}_{\gamma\delta}$.

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