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ON A PROBLEM OF ERDÖS AND IVIĆ

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Abstract. Let us usual $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors and the number of total prime factors of n respectively. Asymptotic formulas for the sum $\sum_{2\le n\le x} n^{-1/\Omega(n)}$ and the logarithm of the sum $\sum_{2\le n\le x} n^{-1/\omega(n)}$ are derived.

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1. Indtroduction and statement of results. Let as usual $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors and the number of total prime factors of n. In [**1**] Erdös and Ivić proved that for $x \ge x_0$ and some constants $0 < c_2 < c_1$, the following inequality holds

$$x\exp\{-c_1(\log x \log_2 x)^{1/2}\} \le \sum_{2\le n\le x} n^{-1/\omega(n)} \le x\exp\{-c_2(\log x \log_2 x)^{1/2}\},$$

where $\log_2 x = \log\log x$. In [**1**] they also proved that for $x \ge x_0$ and some constants $0 < c_4 < c_3$ the inequality

$$x\exp\{-c_3(\log x)^{1/2}\} \le \sum_{2\le n\le x} n^{-1/\Omega(n)} \le x\exp\{-c_4(\log x)^{1/2}\}$$

holds. Then they conjecture that for some $c_5, c_6 > 0$, one has

$$\sum_{2\le n\le x} n^{-1/\omega(n)} = x\exp\{-(c_5+o(1))(\log x\log_2 x)^{1/2}\}, \tag{1.1}$$

and

$$\sum_{2\le n\le x} n^{-1/\Omega(n)} = x\exp\{-(c_6+o(1))(\log x)^{1/2}\}. \tag{1.2}$$

The aim of this note is to give an asymptotic formula for sum in (1.2) and an asymptotic formula for the logarithm of sum in (1.1), respectively. Our results may be stated as follows.

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THEOREM 1.

$$\sum_{2\le n\le x} n^{-1/\Omega(n)} =$$
$$= C\pi^{1/2}(\log 2)^{-3/4}\cdot x(\log x)^{5/4}\cdot \exp\{-2(\log 2\log x)^{1/2}\}(1+O((\log x)^{-1/4})),$$

where

$$C = (1/4)\prod_{p>2}(1+1/(p(p-2)) = 0.378694. \tag{1.3}$$

THEOREM 2.

$$\sum_{2\le n\le x} n^{-1/\omega(n)} =$$
$$= x\exp\left\{-(2\log x\log_2 x)^{1/2}\cdot\left(1-\frac{3\log_3 x}{2\log_2 x}+\frac{3\log 2-2}{2\log_2 x}-\frac{\log_3 x}{\log_2^2 x}+O\left(\frac{1}{\log_2^2 x}\right)\right)\right\}.$$

Thus the above two theorems settle affirmatively the conjecture of Erdös and Ivić in question.

Notations. Let x be a positive variable assumed to be sufficiently large,

$N(x,k) =\mid \{n\le x \mid \Omega(n)=k\}\mid$, $\quad \pi(x,k) =\mid \{n\le x\mid \omega(n)=k\}\mid$

$A = x\exp\{-10(\log x)^{1/2}\}$, $\quad B = x\exp\{-10(\log x\log_2 x)^{1/2}\}$,

$P = (\log x)^{1/2}$, $\quad Q = (\log x)^{1/2}(\log_2 x)^{-1/2}$, $\quad L = (\log x\log_2 x)^{1/2}$,

$M = (\log 2\log x)^{1/2}$, $\quad N = (\log 2)^{-1/2}(\log x)^{1/2}$.

2. Proof of Theorem 1. To prove this we need

LEMMA 1 [**2**]. *For $x\ge 3$ and $5\log_2 x\le k\le \log x/\log 2$ we have uniformly*

$$N(x,k) = C(x/2^k)\log(x/2^k) + O\big((x/2^k)\log_2(3x/2^k)\big),$$

where C is defined by (1.3).

Proof of Theorem 1. We shall use an inequality of Erdös and Ivić [**1**, p. 52]:

$$\sum_{n\le x,\,\Omega(n)\ge k} 1 \ll x\cdot 2^{-k/4}, \quad \text{if} \quad k\ge (\log_2)^2,$$

where the $\ll$-constant is uniform in k and x. From this we have

$$\sum_{n\le x,\ \Omega(n)\ge 40P} n^{-1/\Omega(n)} \ll xe^{-4P}$$

It is evident that

$$\sum_{A<n\le x,\ \Omega(n)\le 0.1P} n^{-1/\Omega(n)} \ll xe^{-4P}.$$

From the above two estimates we get

$$W(x) := \sum_{2\le n\le x} n^{-1/\Omega(n)} = \sum_{\substack{A<n\le x\\ 0{,}1P<\Omega(n)\le 40P}} n^{-1/\Omega(n)} + O(xe^{-4P}) = W' + O(xe^{-4P}),\ \ \textit{say.} \tag{2.1}$$

Lemma 1 implies that

$$N(n,k) = C(n/2^k)\log(n/2^k) + r_n, \ \text{ where } \ r_n = O\big(n/2^k)\log_2(3n/2^k)\big).$$

Therefore we have

$$W' = \sum_{0.1P<k\le 40P}\ \sum_{A<n\le x} n^{-1/k}\big(N(n,k)-N(n-1,k)\big) \tag{2.2}$$
$$= C\sum_{N<k\le 40P} 2^{-k}\sum_{A<n\le x} n^{-1/k}\log n + C\sum_{0.1P<k\le N} 2^{-k}\sum_{A<n\le x} n^{-1/k}\log n$$
$$+ \sum_{0.1P<k\le 40P}\ \sum_{A<n\le x} n^{-1/k}(r_n - r_{n-1}) + O\Big(\sum_{0.1P<k\le 40P} k\cdot 2^{-k}\sum_{A<n\le x} n^{-1/k}\Big)$$
$$= W_1 + W_2 + W_3 + O(W_4),\ \text{ say.}$$

First we estimate W_1 in (2.2). We shall use the following inequality [**3**, Ch. 5, Theorem 8.2]

$$\Big|\sum_{a\le n\le \xi} f(n) - \int_a^\xi f(v)dv - \alpha\Big| \le f(\xi-1),\ \text{ if } \ \xi\ge a+1, \tag{2.3}$$

where $f(x)$ is decreasing, $f(x)\to 0$, if $x\to\infty$ and $0\le\alpha\le f(a)$. Alternatively, we may use the Euler-Maclaurin summation formula.

By this we have

$$\sum_{A<n\le x} n^{-1/k}\log n = \int_A^x z^{-1/k}\log z dz + O(A^{-1/k}\log A)$$
$$= (x\log x)\cdot x^{-1/k}(1+O(P^{-1})).$$

So

$$W_1 = Cx\log x\sum_{N<k\le 40P} 2^{-k}x^{-1/k}(1+O(P^{-1})). \tag{2.4}$$

Using (2.3) again, we have

$$\sum := \sum_{N<k\le 40P} 2^{-k}x^{-1/k} = \int_N^{40P} 2^{-z}x^{-1/z}dz + O(e^{-2M}). \tag{2.5}$$

Setting $z = Nu^2$, we have

$$\sum = N\int_1^\infty \exp\{-M(u^2+u^{-2})\}\cdot 2udu + O(e^{-2M}).$$

Setting further $M^{1/2}(u-u^{-1})=v$, we obtain

$$\sum = Ne^{-2M}\int_0^\infty e^{-v^2}\Big(\frac{v}{M}+\frac{v^2+2M}{M(v^2+4M)^{1/2}}\Big)dv + O(e^{-2M}).$$

Using Lagrange's mean value theorem, we have

$$\mid (v^2+4M)^{-1/2}-(1/2)M^{-1/2}\mid \le (1/2)v^2M^{-3/2},$$

so

$$\begin{aligned}\sum &= NM^{-1/2}e^{-2M}\Big(\int_0^\infty e^{-v^2}dv + O(M^{-1/2}\int_0^\infty e^{-v^2}\cdot vdv)\Big)\\ &= (1/2)\pi^{1/2}NM^{-1/2}e^{-2M}(1+O(P^{-1/2})).\end{aligned} \tag{2.6}$$

Combining (2.4) — (2.6) we get

$$W_1 = (1/2)C\pi^{1/2}(\log 2)^{-3/4}x(\log x)^{5/4}\exp\{-2(\log 2\cdot\log x)^{1/2}\}\cdot (1+O\big((\log x)^{-1/4}\big)). \tag{2.7}$$

In the same way we obtain

$$W_2 = (1/2)C\pi^{1/2}(\log 2)^{-3/4}x(\log x)^{5/4}\exp\{-2(\log 2\cdot\log x)^{1/2}\}\cdot (1+O\big((\log x)^{-1/4}\big)). \tag{2.8}$$

We turn now to estimate W_3 in (2.2). We have

$$\begin{aligned}W_3 &= \sum_{0.1P<k\le 40P}\ \sum_{A<n\le x} n^{-1/k}(r_n-r_{n-1})\\ &= \sum_k\sum_n r_n(n^{-1/k}-(n+1)^{-1/k}+[x]^{-1/k}r_{[x]}-([A]+1)^{-1/k}r_{[A]}\\ &\ll \sum_k\sum_n k^{-1}\cdot 2^{-k}n^{-1/k}\log_2 x + x\sum 2^{-k}x^{-1/k}\log_2 x\\ &\ll (\log x)^{-3/2}\log_2 x\sum_k 2^{-k}\sum_n n^{-1/k}\log n + x\log_2 x\sum_k 2^{-k}x^{-1/k}\\ &\ll (\log x)^{-3/2}\log_2 x\cdot(W_1+W_2)+x\log_2 x\cdot(\textstyle\sum).\end{aligned}$$

Combining (2.6)– (2.8) we get

$$W_3 \ll x(\log x)^{1/4}\log_2 xe^{-2M}. \tag{2.9}$$

It is easy to prove that $W_4 \ll x(\log x)^{3/4}e^{-2M}$. From this and (2.1), (2.2), (2.7), (2.8) and (2.9), the Theorem follows.

3. Proof of Theorem 2. To prove Theorem 2 we shall need the following result on the function $\pi(x,k)$.

LEMMA2. [**4**] *Let* $U = U(x,k) = \log_2 x - \log k - \log_2 k$. *Then the estimate*

$$\pi(x,k) = \frac{x}{k!}\exp\left\{k\left(\log U + \frac{\log U}{U} + O\left(\frac{1}{U}\right)\right)\right\}$$

holds uniformly in the range $x \geq 3$, $(\log_2 x)^2 \leq k \leq \log x/3\log_2 x$.

Proof of Theorem 2. We shall use a classical inequality of Hardy and Ramanujan [**5**, p. 265]:

$$\sum_{n\leq k,\ \Omega(n)=k} 1 < Ex(\log^{-1} x)(\log_2 x + F)^k/k!.$$

where $E, F > 0$ are absolute constants, $x \geq 2$ and $k \geq 0$.

From this we have

$$\begin{aligned}\sum_{n\leq x,\ \Omega(n)\geq k} 1 &\ll x\sum_{r\geq k}\exp\{r\log_3 x - r\log r + O(\gamma)\}\\ &\ll x\exp\{-k\log k + O(k\log_3 x)\}.\end{aligned}$$

So

$$\sum_{n\leq x,\ \omega(n)>10Q} n^{-1/\omega(n)} \ll xe^{-4L},$$

and obviously

$$\sum_{B<n\leq x,\ \omega(n)\leq 0.1Q} n^{-1/\omega(n)} \ll xe^{-4L}.$$

From the above two estimates we can get

$$\begin{aligned}T(x) :&= \sum_{2\leq n\leq x} n^{-1\omega(n)} = \sum_{B<n\leq x,\ 0.1\,Q<k\leq 10Q} n^{-1/k} + O(xe^{-4L})\\ &= \sum_{0.1Q<k\leq 10Q}\exp\{-(1/k)\log x + O(\log_2 x)\}(\pi(x,k) + O(xe^{-2L}))\end{aligned}\tag{3.1}$$

For $0.1Q < k \leq 10Q$, using Lemma 2 and Stirling's formula we get

$$\pi(x,k) = x\exp\{-k(\log k - 1 - \log U - (\log U/U) + O(1/U)).\tag{3.2}$$

From this and (3.1) we deduce

$$\begin{aligned}T(x) =&x\sum_{0.1Q<k\leq 10Q}\exp\{-(1/k)\log x - k(\log k -\\ &- 1 - \log U - (\log U/U) + O(1/U)).\end{aligned}\tag{3.3}$$

Now we prove the upper bound

$$\begin{aligned}T(x) \leq x\exp\Bigg\{&-2(\log x\log_2 x)^{1/2}\left(1 - \frac{3\log_3 x}{2\log_2 x} +\right.\\ &\left.+\frac{3\log 2 - 2}{2\log_2 x} - \frac{\log_3 x}{\log_2^2 x} + O\left(\frac{1}{\log_2^2 x}\right)\right)\Bigg\}.\end{aligned}\tag{3.4}$$

Let

$$F(x,k) = k^{-1}\log x + k(\log k - 1 - \log U - (\log U/U) + O(1/U)),$$

and $a_i = (t + \delta_i D(\log_3 x/\log_2 x))Q$, $(i = 1, 2)$, where $\delta_1 = -1$, $\delta_2 = 1$ and D denotes positive absolute constants, not necessarily the same ones. For $k \in [a_1, a_2]$ we have

$$F(x,k) = ((t/2) + (1/t))(\log x \log_2 x)^{1/2} + O(Q \log_3 x) = G(x,t), \text{ say.}$$

Hence we have

$$\begin{aligned} T(x) &\le \max_{0.1 \le t \le 10} x \exp\{-g(x,t\} \sum_{0.1Q \le k \le 10Q} 1 \\ &\le x \exp\{-(2\log x \log_2 x)^{1/2} + O((\log x))^{1/2}(\log_2 x)^{-1/2} \log_3 x). \end{aligned}$$

Further let $b_i = (2^{1/2} + t_1(\log_3 x/\log_2 x) + \delta_i D(1/log_2 x))Q$, $(i = 1, 2)$.

For $k \in [b_1, b_2]$ we have

$$\begin{aligned} F(x,k) = \left(\frac{1}{2}\log x \log_2 x\right)^{1/2} &\left\{\left(1 + \frac{t_1}{2^{1/2}}\frac{\log_3 x}{\log_2 x} + O\left(\frac{1}{\log_2 x}\right)\right)^{-1} + \right. \\ &\left.\left(1 + \frac{t_1}{2^{1/2}}\frac{\log_3 x}{\log_2 x} - \frac{3\log_3 x}{\log_2 x} + O\left(\frac{1}{\log_2 x}\right)\right)\right\} = G_1(x,t_1), \text{ say.} \end{aligned} \tag{3.5}$$

Hence

$$T(x) \le \max_{-D \le t_1 \le D} x \exp\{-G_1(x,t_1)\} \sum_{0.1Q \le k \le 10Q} 1. \tag{3.6}$$

Let

$$f(t_1) = \left(1 + \frac{t_1}{2^{1/2}}\frac{\log_3 x}{\log_2 x}\right)^{-1} + \frac{t_1}{2^{1/2}}\frac{log_3 x}{\log_2 x};$$

then

$$f'(t_1) = -\left(1 + \frac{t_1}{2^{1/2}}\frac{\log_3 x}{\log_2 x}\right)^{-2}\frac{log_3 x}{2^{1/2}\log_2 x} + \frac{\log_3 x}{2^{1/2}\log_2 x},$$

which means that $f'(t_1) = 0$ for $t_1 = 0$. From this and (3.5), (3.6) we have

$$T(x) \le x \exp\left\{-(2\log x \log_2 x)^{1/2}\left(1 - \frac{3\log_3 x}{2\log_2 x} + O\left(\frac{1}{\log_2 x}\right)\right)\right\}.$$

Proceeding as before, finally let

$$e_1 = (2^{1/2} + t_4(\log_3 x/\log_2^2 x) + \delta_i D(1/\log_2^2))Q, \quad (i = 1, 2).$$

For $k \in [e_1, e_2]$ we have

$$\begin{aligned} F(x,k) = \left(\frac{1}{2}\log x \log_2 x\right)^{1/2} &\left\{\left(1 + \frac{t_4}{2^{1/2}}\frac{\log_3 x}{\log_2^2 x} + O\left(\frac{1}{\log_2^2 x}\right)\right)^{-1} + \right. \\ &\left.\left(1 - \frac{3\log_3 x}{\log_2 x} + \frac{3\log 2 - 2}{\log_2 x} + \left(\frac{t_4}{2^{1/2}} - 2\right)\frac{\log_3 x}{\log_2 x} + O\left(\frac{1}{\log_2^2 x}\right)\right)\right\} \\ &= G_4(x,t_4), \text{ say.} \end{aligned} \tag{3.7}$$

Hence we have

$$T(x) \le \max_{-D \le t_4 \le D} x \exp\{-G_4(x,t_4)\} \sum_{0.1Q \le k \le 10Q} 1. \tag{3.8}$$

Let

$$g(t_4) = \left(1 + \frac{t_4 \log_3 x}{2^{1/2} \log_2^2 x}\right)^{-1} + \frac{t_4 \log_3 x}{2^{1/2} \log_2^2 x};$$

then

$$g'(t_4) = -\left(1 + \frac{t_4 \log_3 x}{2^{1/2} \log_2^2 x}\right)^{-2} + \frac{\log_3 x}{2^{1/2} \log_2^2 x} + \frac{\log_3 x}{2^{1/2} \log_2^2 x},$$

which means that $g'(t_4) = 0$, for $t_4 = 0$. From this and (3.7), (3.8), we can derive (3.4).

To finish the proof of Theorem 2 it remains to show

$$\begin{aligned} T(x) \ge x \exp\biggl\{ &- (2 \log x \log_2 x)^{1/2} \biggl(1 - \frac{3 \log_3 x}{2 \log_2 z} + \\ &+ \frac{3 \log 2 - 2}{2 \log_2 x} - \frac{\log_3 x}{\log_2^2 x} + O\Bigl(\frac{1}{\log_2^2 x}\Bigr)\biggr)\biggr\}. \end{aligned} \tag{3.9}$$

Let $k_0 \le (2 \log x / \log_2 x)^{1/2}$. It is evident that

$$T(x) \ge \sum_{2 \le n \le x,\ \omega(n) = k_0} n^{-1/\omega(n)} \ge x^{-1/k_0} \pi(x, k_0).$$

From this and (3.2) we have

$$\begin{aligned} T(x) \ge x \exp\{ &- (1/k_0) \log x - k_0 (\log k_0 - \\ &- 1 - \log U - (\log U / U) + O(1/U))\} \end{aligned} \tag{3.10}$$

Now putting $k_0 = [(2 \log x / \log_2 x)^{1/2}]$ in (3.10), we can derive (3.9). The proof is completed.

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