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ASYMPTOTIC PROPERTIES OF CONVOLUTION PRODUCTS OF FUNCTIONS

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Abstract. The asymptotic behaviour of convolution products of the form $\int_0^x f(x-y)g(y)dy$ is studied. From our results we obtain asymptotic expansions of the form

$$R(x) := \int_o^x f(x-y)g(y)dy - f(x)\int^\infty g(y)dy - g(x)\int_0^\infty f(y)dy = O(m(x)).$$

Under rather mild conditions on f, g and m the O-term can be calculated more explicitely as

$$R(x) - (f(x-1)-f(x))\int_0^\infty yg(y)dy + (g(x-1)-g(x))\int_0^\infty yf(y)dy + o(m(x)).$$

An application in probability theory is included.

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1. Introduction. In a recent series of papers, several authors have studied the asymptotic bahaviour of the convolution product

$$f * g(x) := \int_0^x f(x-y)\ g\ (y)dy \quad (x \geq 0)$$

for functions f and g in a suitable class of functions. In [**6**] Luxemburg introduced the following class Λ of "admissible" functions:

A function L belongs to the class Λ if it is continuous and if

$$(1.1) \qquad \begin{aligned} &(i)\ L(x+h) \sim L(x) \qquad (x \to \infty), \text{ for all } h \in R \\ &(ii)\ \sup_{x>0} \max_{x \leq t \leq 2x} L(t)/L(2x) < \infty. \end{aligned}$$

Using this class of functions Luxemburg estimates $f * g(x)$ as follows.

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LEMMA 1.1. [**6**, Theorem 2.2] *If $f, g \in L[0,\infty)$ and if $L \in \Lambda$ such that as $x \to \infty$ $f(x) \sim F \cdot L(x)$ and $g(x) \sim G \cdot L(x)(F, G \in R)$, then as $x \to \infty$*

$$f * g(x) \sim \left(F \int_0^\infty g(y)dy + G \int_0^\infty f(y)dy\right) L(x). \blacksquare$$

In [**5**] Geluk shows that the difference betwen $f * g(x)$ and a suitable linear combination of $f(x)$ and $g(x)$ is small compared to $f * g(x)$, $f(x)$ and $g(x)$.

LEMMA 1.2. [**5**, Theorem 2] *If $f, g \in L^1[0,\infty)$ belong to the class Λ_0 defined below and if for some $c > 0$, $L(x) := f(x-1) - f(x) \sim c(g(x-1) - g(x))$ $(x \to \infty)$, then as $x \to \infty$,*

$$\left(f * g(x) - f(x) \int_o^\infty g(y)dy - g(x) \int_0^\infty f(y)dy\right) \sim \left(\int_0^\infty t(c^{-1}f(t) + g(t))dt\right) L(x). \blacksquare$$

The class Λ_0 used in Lemma 1.2 is defined as follows: a positive function g belongs to the class Λ_0 if $g \in L^1[0,\infty)$ and if

- (i) $L(x) := g(x-1) - g(x)$ is positive for all x sufficiently large;
- (ii) $\lim_{x\to\infty} (g(x+a) - g(x))/L(x) = -a$, $\forall a \in \mathbf{R}$
- (iii) $\limsup_{x\to\infty} g(2x)/g(x) < 1$
- (iv) $\limsup_{x\to\infty} \sup_{x \le t \le 2x} \left| \dfrac{g(t) - g(2x)}{(2x - t)L(2x)} \right| < \infty.$

As an exemple of Λ_0 we mention the function $g(x) = x^{-\alpha}(\ln x)^\beta$ for $x > 1$ and $g(x) = 0$ for $x \le 1$, where $\alpha > 2$ and $\beta \in \mathbf{R}$. Note that condition (iv) implies that $g'(x)$ exists and that

$$\limsup_{x\to\infty} \mid g'(x) \mid /L(x) < \infty. \tag{1.2}$$

Also note that (ii) implies $L(x + a) \sim L(x)(x \to \infty)$ for $a \in \mathbf{R}$, i. e. L satisfies (1.1).

In this paper we plan to extend the results of Lemmas 1.1 and 1.2 and to this end we will consider the class of functions $\Lambda(m)$ defined by

$$\Lambda(m) = \{g : \mathbf{R}^+ \to \mathbf{R}^+ \mid \sup_{x \ge 0} \mid g'(x) \mid /m(x) < \infty\}$$

where the auxiliary function m belongs to some suitable class of functions. We also give an application of our results in probability theory. This application is an improvement of an estimate for the tail of the distribution function of the $n-$fold convolution of a random variable, see Feller [**4**, VIII] and Geluk [**5**, p. 88]. Before stating our main results it should be remarked that the results below are comparable to the results obtained in [**7**] where the convolution product of sequences is considered.

2. Main results. In order to estimate the asymptotic behaviour of the convolution product $f * g$ we first recall some classes of functions which will be frequently used in the following. The class $\mathcal{L}$ is the class of measurable functionc $f:\mathbf{R}_+\to\mathbf{R}_+$ such that

$$\lim_{x\to\infty} f(x+y)/f(x)=1,\quad \forall y\in\mathbf{R}. \tag{2.1}$$

If $f\in\mathcal{L}$, then the convergence in (2.1) is uniform in y-compact subsets of $\mathbf{R}$, see [**1**, I]. The following class SD has been introduced by Chover et. al [**2**]. A continuous function $m:\mathbf{R}_+\to\mathbf{R}_+$ is in the class SD if $m\in\mathcal{L}\cap L[0,\infty)$ and if

$$\lim_{x\to\infty} m*m(x)/m(x)=c \tag{2.2}$$

It is known that for $m\in SD$ the constant c in (2.2) equals $c=2\int_0^\infty m(y)dy$. This result is by no means trivial and its proof heavily depends on Banach algebra techniques, see [**2**].

We first prove the following best possible extension of Lemma 1.1.

THEOREM 2.1. *Suppose* $\mid f\mid,\mid g\mid\in L[0,\infty)$ *and* $m\in SD$ *are such that* $f(x)\sim Fm(x)$ *and* $g(x)\sim Gm(x)\,(F,G\in\mathbf{R})$*, then*

$$f*g(x)\sim\left(F\int_0^\infty g(y)dy+G\int_0^\infty f(y)dy\right)m(x).\qquad\Box$$

Remark. Here and in the following we will use the notation $f(x)\sim ag(x)$ to abbreviate $\lim_{x\to\infty} f(x)/g(x)=a$.

Proof. Under the conditions of the theorem there exist constants M and $x_0>0$ such that for $x\ge x_0$.

$$|f(x)|\le Mm(x)\ \text{ and }\ |g(x)|\le Mm(x) \tag{2.3}$$

For $x\ge 2x_o$ we now have

$$f*g(x)=\left\{\int_0^{x_0}+\int_{x_0}^{x-x_0}+\int_{x-x_0}^{x}\right\}f(x-y)\,g(y)dy=:\text{I+II+III}.$$

Since $m\in SD$ implies that $m(x-y)\sim m(x)\,(x\to\infty)$ uniformly for $y\in[0,x_0]$ we have

$$\lim_{x\to\infty}\frac{I}{m(x)}=\lim_{x\to\infty}\int_0^{x_0}\frac{f(x-y)}{m(x-y)}\frac{m(x-y)}{m(x)}g(y)dy=F\int_0^{x_0}g(y)dy \tag{2.4}$$

and similary we have

$$\lim_{x\to\infty}\text{III}/m(x)=G\int_0^{x_0}f(y)dy. \tag{2.5}$$

Finally, using (2.3) we have

$$\mid\text{II}\mid\le M^2\int_{x_0}^{x-x_0}m(x-y)m(y)dy.$$

Using $m \in SD$ and hence $m \in \mathcal{L}$ we have

$$\lim_{x\to\infty} \frac{1}{m(x)} \int_{x_o}^{x-x_0} m(x-y)m(y)dy = \lim_{x\to\infty} \frac{m*m(x)}{m(x)} - 2\lim_{x\to\infty}\int_0^{x_0} \frac{m(x-y)}{m(x)} m(y)dy$$
$$= 2\int_{x_0}^{\infty} m(y)dy$$

and it follows that

$$\lim_{x\to\infty} \sup \mid II \mid /m(x) \le M^2 2 \int_{x_0}^{\infty} m(y)dy \tag{2.6}$$

Now combine (2.4)—(2.6) and let $x_0 \to \infty$ to obtain the proof of the theorem. ■

Remark 2.2. The following O-analogue of Theorem 2.1. is clear: if $f(x) = O(m(x))$, $g(x) = O(m(x))$ and $m*m(x) = O(m(x))$, then also $f*g(x) = O(m(x))$. ■

Since verifying $m \in SD$ may be difficult we now introduce the classes $\mathcal{D}$ and $\mathcal{R}(\alpha)$. A measurable function $f : \mathbf{R}_+ \to \mathbf{R}_+$ is in the class $\mathcal{D}$ of function of dominated variation if

$$\limsup_{x\to\infty} f(xy)/f(x) < \infty, \ \ \forall y > 0. \tag{2.7}$$

The function f is regulary varying with index $\alpha (=: f \in \mathcal{R}(\alpha))$ if

$$\lim_{x\to\infty} f(xy)/f(x) = y^\alpha, \ \ \forall y > 0. \tag{2.8}$$

Whenever $f \in \mathcal{D}$ or $f \in \mathcal{R}(\alpha)$ the convergence in (2.7) or (2.8) is uniform in y-compact subsets of $\mathbf{R}_+$, see [**1**, I]. It is well known that $\mathcal{R}(\alpha) \subset \mathcal{D} \cap \mathcal{L}$ and that $\mathcal{D} \cap \mathcal{L} \cap L[0,\infty) \subset SD$, see e. g. [**1, 3**]. From this it is clear that the class Λ of admissible $L[0,\infty)$-functions is a subclass of SD so that Lemma 1.1 is implied by Theorem 2.1. Another consequence of Theorem 2.1. is the following

Corollary 2.3. *If $f \in SD$, then $f^{*n} \in SD$ and $f^{*n}(x) \sim nf(x) \left(\int_0^\infty f(y)dy\right)^{n-1}$.* ■.

To obtain the rate of convergence in Corollary 2.3, Theorem 2.1 may be used again. Suppose $f \in L[0,\infty)$ with $\int_0^\infty f(y)dy = 1$ and for $n \ge 2$ define $r_n(x) := f^{*n}(x) - nf(x)$.

Lemma 2.4. *If $m \in SD$ and if $r_2(x) \sim Am(x)$ and $r_3(x) \sim Bm(x)$ where $A, B \in \mathbf{R}$, then for all $n \ge 2$*

$$\lim_{x\to\infty} r_n(x)/m(x) = Bn(n-1)(n-2)/6 - An(n-1)(n-3)/2.$$

Proof. For $n \ge 2$ we have

$$r_{n+2}(x) = r_n * r_2(x) + 2r_{n+1}(x) + nr_3(x) - 2nr_2(x).$$

Using $\int_0^\infty r_n(y)dy = 1 - n$ it follows from Theorem 2.1 and by induction that for all $n \geq 2$

$$\lim_{x\to\infty} r_{n+2}(x)/m(x) = : c_{n+2}$$

exists and that

$$c_{n+2} = 2c_{n+1} - c_n + n(c_3 - 3c_2) + c_2.$$

Solving this difference equation and using $c_2 = A,\ \ e_3 = B$ we obtain the expression for c_n. $\square$

Remark 2.5. The following O-analogue of Lemma 2.4 is obvious: if $r_2(x) = O(m(x)),\ \ r_3(x) = O(m(x))$ and $m * m(x) = O(m(x))$, then for all $n \geq 2,\ r_n(x) = O(m(x))$. $\blacksquare$

The difficulty in Lemma 2.4 of course lays in checking the condition on r_2 and r_3. In the next result we shall restrict ourselves to the class $\Lambda(m)$ of functions defined as (*cf.* (1.2)):

$$\Lambda(m) = \{g : \mathbf{R}_+ \to \mathbf{R}_+ \mid K(g) : = \sup_{x\geq 0} \mid g'(x) \mid /m(x) < \infty\}$$

Here m belongs to some class of functions to be specified later. If $m \in L[0,\infty)$ we define $M(x) : = \int_o^\infty m(s)ds$ and w.l.o.g we assume that $M(\infty) = 1$. Also define $M_1(x)$ as $M_1(x) : = 1 - M(x)$.

For functions $g \in \Lambda(m)$ with $m \in L[0,\infty)$ the following inequalites are often useful: for $x \geq y \geq 0$

$$\mid g(x) - g(y) \mid \leq \int_y^x \mid g'(s) \mid ds \leq K(g)(M(x) - M(y)) \tag{2.9}$$

$$\mid g(x) \mid \leq \int_x^\infty \mid g'(s) \mid ds \leq K(g)M_1(x). \tag{2.10}$$

We first prove that under some mild condition on $m(x),\ \Lambda(m)$ is closed under $*$.

Lemma 2.6. *If $m \in L[0,\infty)$ and if as $x \to \infty$,*

$$M_1^2(x) = O(m(x)) \tag{2.11}$$

$$1 - M * m(x) - 2M_1(x) = O(m(x)) \tag{2.12}$$

then $\Lambda(m)$ is closed under $*$. $\blacksquare$

Proof. Assume $f, g \in \Lambda(m)$ and consider $(f * g)'(x)$. We have

$$(f * x)'(x) = \int_0^x f'(x-y)(g(y) - g(x))dy + g(x)f(x).$$

Using (2.9) and (2.10) for f or g we obtain

$$\mid (f * g)'(x) \mid \leq K(f)K(g)\bigg(\int_0^x m(x-y)(M(x) - M(y))dy + M_1^2(x)\bigg).$$

Since

$$\int_o^x m(x-y)(M(x)-M(y))dy = M^2(x)-m*M(x) = 1-m*M(x)-2M_1(x)+M_1^2(x)$$

we obtain that $|(f*g)'(x)| = O(m(x))$.■

In our next result we estimate the difference

$$R(x) := f*g(x) - f(x)\int_0^\infty g(y)dy - g(x)\int_0^\infty f(y)dy$$

under various conditions on m, f and g. We start with an O-type of result.

THEOREM 2.7. *Assume* $m \in L^1[0,\infty)$ *and* $g \in \Lambda(m)$.

(i) If $f(x) = O(m(x))$ *and if* (2.11) *and* (2.12) *hold, then* $R(x) = O(m(x))$.

(ii) If $f \in \Lambda(m)$ *and if as* $x \to \infty$

$$M_1 * M_1(x) - 2M_1(x)\int_0^\infty M_1(y)dy = O(m(x)) \tag{2.13}$$

$$xM_1^2(x) + M_1(x)\int_x^\infty M_1(y)dy = O(m(x)) \tag{2.14}$$

then $R(x) = O(m(x))$. □

Proof. (i) Let $M := \sup |f(x)|/m(x)$ and write $R(x)$ as

$$R(x) = \int_0^x f(y)(g(x-y)-g(x))dy - g(x)\int_x^\infty f(y)dy - f(x)\int_o^\infty g(y)dy.$$

It follows from (2.9) and (2.10) that

$$|R(x)| \le MK(g)\left(\int_0^x m(y)(M(x)-M(x-y))dy + M_1^2(x) + m(x)\int_x^\infty M_1(y)dy\right).$$

The result now follows as in the proof of Lemma 2.6.

(ii) Now we use the decomposition

$$\begin{aligned} R(x) &= \int_0^\infty (f(x-y)-f(x))(g(y)-g(x)dy \\ &\quad -\left(f(x)\int_x^\infty g(y)dy + g(x)\int_x^\infty f(y)dy + xf(x)g(x)\right) \\ &=: \text{I-II} \end{aligned}$$

Using (2.10) we have

$$|\,\text{II}\,| \le K(f)K(g)\left(2M_1(x)\int_x^\infty M_1(y)dy + xM_1^2(x)\right) \tag{2.15}$$

which by assumption is $O(m(x))$ as $x \to \infty$. As to I, using (2.9) we have

$$| \mathrm{I} | \leq K(f)k(g) \int_0^x (M(x) - M(x-y))(M(x) - M(y))dy$$

so that

$$| \mathrm{I} | \leq K(f)K(g)\Big(M_1 * M_1(x) - 2M_1(x)\int_0^\infty M_1(y)dy + 2M_1(x)\int_x^\infty M_1(y)dy + xM_1^2(x)\Big).$$

Using (2.13) and (2.14) we also obtain that $| I | = O(m(x))$ and the proof of the Theorem.■

Corollary 2.8. *If $f \in \Lambda(m)$ with $\int_0^\infty f(y)dy = 1$ and if m satisfies the hypothesis of Theorem* 2.7. (i) *and* (ii), *then for all $n \geq 2, r_n(x) = O(m(x))$.*

Proof. It follows from Lemma 2.6 that $f^{*n} \in \Lambda(m)$ for all $n \geq 2$. Also, from the definition of r_n it follows that for all $n \geq 2$

$$r_{n+1}(x) = r_n * f(x) - (1-n)f(x) + nr_2(x). \tag{2.16}$$

Now Theorem 2.7 (ii) gives $r_2(x) = O(m(x))$; by induction it follows from Theorem 2.7 (ii) and (2.16) that for all $n \geq 2$, also $r_{n+1}(x) = O(m(x))$.■

If more is assumed about m od M_1 we prove that $R(x)$ asymptotically equals a constant times $m(x)$. For further use we define

$$\Lambda^G(m) : = \{\in \Lambda \mid \lim_{x\to\infty} (g(x-y) - g(x))/m(x) = Gy, \ \ \forall y \in \mathbf{R}\}$$

Note that if $m \in \mathcal{L}$ and $g \in \Lambda^G(m)$, then $\lim_{x\to\infty} (g(x-y) - g(x))/m(x) = Gy$ holds uniformly in y-compact of $\mathbf{R}$.

Theorem 2.9. *Let $m \in \mathcal{L} \cap L^1[0,\infty)$ and $g \in \Lambda^G(m)$.*

(i) *If $f(x) \sim Fm(x) \quad (x \to \infty, F \in \mathbf{R})$ and if*

$$\lim_{x\to\infty} (1 - M * m(x) - 2M_1(x))/m(x) = c \tag{2.17}$$

and

$$M_1^2(x) = o(m(x)) \quad (x \to \infty) \tag{2.18}$$

then $c = 2\int_0^\infty ym(y)dy$ and

$$\lim_{x\to\infty} R(x)/m = G \int_0^\infty yf(y)dy. \tag{2.19}$$

(ii) *If $f \in \Lambda^F(m)$, if $m \in \mathcal{L} \cap L^2[0,\infty)$ and if*

$$\lim_{x\to\infty} \frac{1}{m(x)}\left[M_1 * M_1(x) - 2M_1(x)\int_0^\infty M_1(y)dy\right] = c \tag{2.20}$$

and

$$xM_1^2(x) + M_1(x)\int_0^\infty M_1(y)dy = o(m(x)) \quad (x \to \infty) \tag{2.21}$$

then $c = 2\int_0^\infty yM_1(y)dy$ *and*

$$\lim_{x\to\infty} R(x)/m(x) = F\int_0^\infty yg(y)dy + G\int_0^\infty yf(y)dy. \qquad \square \tag{2.22}$$

Proof. (i) We first prove that $c = 2\int_0^\infty ym(y)dy$. To this end note that $(M_1 * M_1)'(x) = -(1 - m * M(x) - 2M_1(x))$. It follows from (2.17) and l'Hopital's rule that $M_1 * M_1(x) \sim cM_1(x)$. Since $m \in \mathcal{L}$ we also have $M_1 \in \mathcal{L}$, whence $M_1 \in SD$ and $c = 2\int_0^\infty M_1(y)dy = 2\int_0^\infty ym(y)dy$.

To prove (2.19), for $x_0 > 0$ and $x \geq 2x_0$ we write $R(x)$ as

$$\begin{aligned} R(x) = & \left(\int_0^{x_0} + \int_{x_0}^{x-x_0} + \int_{x-x_o}^{x}\right) f(y)(g(x-y) - g(x))dy \\ & - g(x)\int_x^\infty f(y)dy - f(x)\int_0^\infty g(y)dy \\ =: & \text{ I+II+III-IV-V} \end{aligned}$$

First consider IV; using (2.10) and (2.18) we have

$$IV = o(m(x)) \quad (x \to \infty). \tag{2.23}$$

By assumption we also have

$$V \sim F\int_0^\infty g(y)dy\, m(x). \tag{2.24}$$

Next consider I; since $g \in \Lambda^G(m)$ implies that $g(x-y) - g(x) \sim Gym(x)$ uniformly for $y \in [0, x_0]$ we have

$$I \sim G\int_0^{x_0} yf(y)dy\, m(x). \tag{2.25}$$

As to II, as in the proof of Theorem 2.7 (i) we have

$$| II | \leq MK(g)\int_{x_0}^{x-x_0} m(y)(M(x) - M(x-y))dy.$$

Now

$$\begin{aligned} \int_{x_0}^{x-x_0} m(y)(M(x)) - M(x-y))dy = & (1 - M * m(x) - 2M_1(x) - M_1^2(x)) \\ & - \int_0^{x_0} m(y)(M(x) - M(x-y))dy - \int_{x-x_0}^{x} m(y)(M(x) - M(x-y))dy \\ & =: A - B - C \end{aligned}$$

Using (2.17), (2.18) and the value of c we have

$$\lim_{x\to\infty} A/m(x) = 2\int_0^\infty ym(y)dy$$

As to B, since $m \in \mathcal{L}$ we have $M \in \Lambda^1(m)$ whence

$$\lim_{x\to\infty} B/m(x) = \int_0^\infty ym(y)dy$$

and using $m \in \mathcal{L}$ once more we have

$$\lim_{x\to\infty} C/m(x) = \int_0^{x_0} (1-M(y))dy.$$

Combining these estimates we obtain

(2.26)

$$\limsup_{x\to\infty} \mid \mathrm{II} \mid /m(x) \leq MK(g)\left(2\int_0^\infty ym(y)dy - \int_0^{x_0} ym(y)dy - \int_0^{x_0}(1-M(y))dy\right).$$

Finally consider III; using $m \in \mathcal{L}$ we have

$$\lim_{x\to\infty}\frac{\mathrm{III}}{m(x)} = \lim_{x\to\infty}\int_0^{x_0}\frac{f(x-y)}{m(x-y)}\frac{m(x-y)}{m(x)}(g(y)-g(x))dy = F\int_0^{x_0} g(y)dy \tag{2.27}$$

An interchange of limit and integral is indeed possible, since $m \in \mathcal{L}$ and

$$\frac{1}{m(x)}\sup_{0\leq y\leq x_0} \mid f(x-y)(g(y)-g(x)) \mid \leq M \sup_{0\leq y\leq x_0}\frac{m(x-y)}{m(x)}\cdot K(g)(1-M(y))$$

and since $\int_0^{x_0}(1-M(y))dy < \infty$. Now combine the estimates (2.23) — (2.27) and let $x_0 \to \infty$ to obtain (2.19).

(ii) We first prove that $c = 2\int_0^\infty yM_1(y)dy$; to this end consider $M_2(x) := \int_0^x M_1(y)dy$ and $M_3(x) = M_2(\infty) - M_2(x)$. We have

$$(M_3 * M_3)''(x) = M_1 * M_1(x) - 2M_1(x)M_2(\infty).$$

Using (2.20) and de l'Hopital's rule we obtain that $M_3 * M_3(x) \sim cM_3(x)$. Since $m \in \mathcal{L}$ implies that $M_3 \in \mathcal{L}$ we obtain that $M_3 \in SD$ and hence that $c = 2\int_0^\infty M_3(y)dy = 2\int_0^\infty yM_1(y)dy$. To prove the Theorem we use the decomposition of $R(x)$ as in the proof of Theorem 2.7 (ii). From (2.15) and (2.21) we obtain

$$\mathrm{II} = o(m(x)) \quad (x\to\infty). \tag{2.28}$$

Next we consider I and for $x \geq 2,\ x_0 > 0$ we write

$$\begin{aligned} I &= \left(\int_0^{x_0} + \int_{x_0}^{x-x_0} + \int_{x-x_o}^{x}\right)(f(x-y)-f(x))(g(y)-g(x))dy \\ &=: A + B + C \end{aligned}$$

As to I, we have $f \in \Lambda^F(m)$ and $\lim_{x\to\infty} g(y) - g(x) = g(y)$ uniformly for $y \in [0, x_0]$, hence

$$\lim_{x\to\infty} A/m(x) = F\int_0^{x_0} yg(y)dy.$$

In a similar way we obtain that

$$\lim_{x\to\infty} C/m(x) = G\int_0^{x_0} yf(y)dy.$$

Next for B we have

$$\mid B \mid \leq K(f)K(g)\int_{x_0}^{x-x_0}(M(x)-M(x-y))(M(x)-M(y))dy.$$

Using (2.20), (2.21) and similar arguments as before we obtain

$$\limsup_{x\to\infty} B/m(x) \leq K(f)K(g)\left[2\int_0^\infty yM_1(y)dy - 2\int_0^\infty yM_1(y)dy\right].$$

Now combine the estimates for A, B and C and let $x_0 \to \infty$ to obtain

$$\lim_{x\to\infty} I/m(x) = F\int_0^\infty yg(y)dy + G\int_0^\infty yf(y)dy$$

and hence the proof of (2.22).■

Remark 2.10. 1) Under the conditions on m of Theorem 2.9 we also have $f * g \in \Lambda^L(m)$. In the case of Theorem 2.9 (i) we have $L = G\int_0^\infty f(z)dz$ and in the other case we have

$$L = G\int_0^\infty f(z)dz + F\int_0^\infty g(z)dz.$$

2) It follows from (2.19) or (2.22) that $R(x)$ can be expanded as follows:

$$R(x) = (f(x-1)-f(x))\int_0^\infty yg(y)dy + (g(x-1)-g(x))\int_0^\infty yf(y)dy + o(m(x))$$■

COROLLARY 2.11. *If $f \in \Lambda^F(m)$ with $\int_0^\infty f(y)dy = 1$ and if m satisfies the hypothesis of Theorem* 2.9 (i) *and* (ii), *then for all $n \geq 2$,*

$$\lim_{x\to\infty} r_n(x/m(x) = n(n-1)F\int_0^\infty yf(y)dy.$$

Proof. If $f \in \Lambda^F(m)$ it follows from Theorem 2.9 (ii) that

$$\lim_{x\to\infty} r_2(x)/m(x) = 2F\int_0^\infty yf(y)dy.$$

Now assume that for some n we have

$$\lim_{x\to\infty} r_n(x)/m(x) = c_n \tag{2.29}$$

it then follows from Theorem 2.9 (i) (with $g \equiv f$ and $f \equiv r_n$) that

$$\lim_{x\to\infty} \frac{1}{m(x)} \left[r_n * f(x) - r_n(x) - f(x) \int_0^\infty r_n(y)dy \right] = F \int_0^\infty y r_n(y) dy.$$

Using (2.29), $\int_0^\infty r_n(y)dy = 1-n$ and $\int_0^\infty y r_n(y)dy = 0$ we obtain that

$$\lim_{x\to\infty} \left[r_n * f(x) - (1-n) f(x) \right] / m(x) = c_n.$$

Using (2.16) and (2.29) we obtain that (2.29) holds with n replaced by $n+1$ and that $c_{n+1} = c_n + nc_2$. Solving this difference equation and using $c_2 = 2F \int_0^\infty y f(y) dy$ we obtain the desired result. ■

In our next result we show that the conditions of the previous results hold for many functions in the classes $\mathcal{D}$ or $\mathcal{R}(\alpha)$. If $m \in \mathcal{D}$, the upper Matuszewska index $\alpha(m)$ of m is defined as [**1**]

$$\alpha(m) := \lim_{x\to\infty} \frac{1}{\log y} \left[\log \limsup_{x\to\infty} \frac{m(xy)}{m(x)} \right].$$

If $m \in \mathcal{R}(\alpha)$ it is clear that $\alpha(m) = \alpha$; if $m \in \mathcal{D}$ then $\alpha(m)$ always exists (possibly $\alpha(m) = \infty$) and if $\alpha(m) < \infty$ then [**1**] for every $\alpha > \alpha(m)$ there exist constants C and x_o such that

$$m(xy)/m(x) \le C y^\alpha, \quad \forall y \ge 1, \quad \forall x \ge x_0. \tag{2.30}$$

We now prove

Proposition 2.12. (i) *If* $m \in \mathcal{D} \cap L^1[0,\infty)$ *with* $\alpha(m) \le -2$, *then the conditions of Lemma* 2.6 *hold. Furthermore, if also* $m \in \mathcal{L}$, *then the conditions of Theorem* 2.9 (i) *hold.*

(ii) *If* $m \in \mathcal{R}(\alpha) \cap L^1[0,\infty)$ *with* $\alpha \le -2$, *then the conditions of Theorem* 2.9 (i) *hold.*

Proof. (i) Choose α such that $\alpha(m) < \alpha < -1$; integrating (2.30) w. r. to y we obtain

$$M_1(x)/xm(x) \le C/(-1-\alpha) =: D, \quad \forall x \ge x_0 \tag{2.31}$$

Hence

$$\frac{M_1^2(x)}{m(x)} \le D x M_1(x) \le D \int_x^\infty y m(y) dy$$

so that (2.11) and (2.18) hold. Next consider (2.12); we have

$$1 - M * m(x) - 2M_1(x) = \int_0^\infty m(y) M(x) - (M(x-y))dy - M_1^2(x) =: \text{I-II}$$

We already proved that II$= o(m(x))(x \to \infty)$ and as to I we write

$$I = \int_0^{x/2} \int_{x-y}^{x/2} m(z)dz \; m(y)dy + \int_0^{x/2} m(x-y)(M(x) - M(y))dy.$$

Since $m \in \mathcal{D}$ we have

$$\sup_{x \geq x_1} \sup_{x/2 \leq z \leq x} m(z)/m \leq M \tag{2.32}$$

for some constants x_1 and M. Hence for $x \geq x_1$

$$I/m(x) \leq M\left(\int_0^{x/2} ym(y)dy + \int_0^{x/2}(M(x) - M(y))dy\right) \leq 2M\int_0^{\infty} ym(x)dy$$

and (2.12) follows. On the other hand, if also $m \in \mathcal{L}$ it follows from Lebesgue'e theorem and (2.32) that

$$\lim_{x\to\infty} \mathrm{I}/m(x) = 2\int_0^{\infty} ym(y)dy$$

so that in this case (2.17) holds.

(ii) If $m \in \mathcal{R}(\alpha)(\alpha \leq -2)$ it follow from Karamata's theorem that (2.31) can be replaced by $M_1(x) \sim xm(x)/(-1-\alpha)$. Now the remainder of the proof of the result is the same as in (i) since $\mathcal{R}(\alpha) \subset \mathcal{D} \cap \mathcal{L}$.■

Proposition 2.13. (i) *If* $m \in \mathcal{D} \cap L^2[0,\infty)$ *with* $\alpha(m) \leq -3$, *then the conditions of Theorem* 2.7 (ii) *hold. Furthermore if* $m \in \mathcal{L}$ *then the conditions of Theorem* 2.9 (ii) *hold.*

(ii) *If* $m \in \mathcal{R}(\alpha) \cap L^2[0,\infty)$ *with* $\alpha \leq -3$, *then the conditions of Theorem* 2.9 (ii) *hold.* □

Proof. Choose α such that $\alpha(m) < \alpha < -2$; integrating (2.30) w. r. to y we obtain (2.31) and integrating once again we obtain for some constant E that

$$\int_x^{\infty} M_1(y)dy \leq Ex^2 m(x). \tag{2.33}$$

From (2.31) it follows that $xM_1^2(x)/m(x) \leq Dx^2 M_1(x) \leq D\int_x^{\infty} y^2 m(y)dx$ and from (2.33) that

$$\frac{1}{m(x)}\left[M_1(x)\int_x^{\infty} M_1(y)dy\right] \leq E\int_x^{\infty} y^2 m(y)dy.$$

Hence (2.21) and (2.14) follow at once. Next we consider $M_1 * M_2(x) - 2M_1(x)$ $\int_0^{\infty} M_1(y)dy$ which can be written as

$$2\left(\int_0^{x/2}(M_1(x-y) - M_1(x))M_1(y)dy - M_1(x)\int_{x/2}^{\infty} M_1(y)dy\right) =: 2(I - II)$$

We already proved that $II = o(m(x))(x \to \infty)$ so that only I has to be considered. Using (2.32) we have

$$M_1(x-y) - M_1(x) = \int_{x-y}^{x} m(s)ds \leq Mm(x)y \tag{2.34}$$

whence $I/m(x) \leq M \int_0^\infty yM_1(y)dy < \infty$ and (2.13) holds. If it is also knowm that $m \in \mathcal{L}$ we consider I; since $m \in \mathcal{L}$ we have

$$\frac{M_1(x-y) - M_1(x)}{m(x)} = \int_{x-y}^{x} \frac{m(s)ds}{m(x)} \rightarrow \quad (x \rightarrow \infty)$$

and using (2.34) and Lebesgue's theorem on dominated convergence it follows that

$$\lim_{x\to\infty} I/m(x) = \int_0^\infty yM_1(y)dy$$

and hence (2.20) holds.

(ii) If $m \in \mathcal{R}(\alpha),\ \alpha \leq -3$, it follows from Karamata's theorem that

$$M_1(x) \sim xm(x)/(-1-\alpha) \text{ and } \int_x^\infty M_1(y)dy \sim xM_1(x)/(-2-\alpha).$$

Hence also here (2.21) holds. The remainder of the proof can now be copied from that of (i) since $\mathcal{R}(\alpha) \subset \mathcal{D} \cap \mathcal{L}$. ■

Combining Corollary 2.11 and Propositions 2.12 and 2.13 we have

COROLLARY 2.14. *Suppose* $m \in \mathcal{D} \cap \mathcal{L} \subset L^2[0,\infty)$ *with* $\alpha(m) \leq -3$ *or* $m \in \mathcal{R}(\alpha) \cap L^2[0,\infty)$ *with* $\alpha \leq -3$. *If* $f \in \Lambda^F(m)$ *with* $\int_0^\infty f(y)dy = 1$, *then for each* $n \geq 2$, $f^{*n} \in \Lambda^{nF}(m)$ *and*

$$\lim_{x\to\infty}(f^{*n}(x) - nf(x))/m(x) = n(n-1)F\int_0^\infty yf(y)dy. ■$$

Corollary 2.14 can be interpreted in probability theory as follows. Let $X, X_1, X_2, \ldots, X_n$ be i.i.d. positive random variables with distribution function (d.f.) G, with density f and finite mean μ. Then $X_1 + \ldots + X_n$ has d.f. $G^{xn}(x)$, the n-fold convolution of G, and has density f^{*n}. If f and m satisfy the hypothesis of Corollary 2.14 we have

$$f^{*n}(n) - nf(x) \sim 2\binom{n}{2}\mu F m(x)$$

and hence

$$1 - G^{xn}(x) - n(1-G(x)) \sim 2\binom{n}{2}\mu F M_1(x). \tag{2.35}$$

The result extends a result of Feller's book [**4**, VIII] which states that

$$1 - G^{xn}(x) \sim n(1-G(x))$$

and (2.35) aslo extends Theorem 4 of Geluk [**5**].

A further extension of (2.35) can be obtained as follows. If $n = 2$ in (2.35) we obtain $1 - G^{x2}(x) - 2(1-G(x)) \sim 2\mu F M_1(x)$; hence to obtain a second-order result in (2.35) it will be convenient to define for $n \geq 3$ the functions $S_n(x)$ as follows:

$$S_n(x) := 1 - G^{xn}(x) - n(1-G(x)) - \binom{n}{2}(1 - G^{x2}(x)) - 2(1-G(x)).$$

Also define $g_n(x)$ as

$$g_n(x) := \sum_{i=1}^{n-3} \binom{n-i-2}{2} f^{*i}(x).$$

We now prove

LEMMA 2.15. *For $n \geq 4$ we have*

$$S_n(x) = \binom{n-1}{2} S_3(x) + S_3 * g_n(x). \tag{2.36}$$

Proof. Since $G^{xi}(x) = \int_0^x g^{*i}(s)ds$ we have

$$S_3 * f^{*i}(x) = G^{xi}(x) - 3G^{xi+1}(x) + 3G^{xi+2}(x) - G^{xi+3}(x).$$

Some straightforward calculations then give

$$S_3 * g_n(x) = -G^{xn}(x) + \binom{n-2}{2} G(x) - (n-3)(n-1)G^{x2}(x) + \binom{n-1}{2} G^{x3}(x)$$

and the Lemma follows.■

It is not hard to show that

$$\int_0^\infty S_3(y)dy = \int_0^\infty yS_3(y)dy = 0 \tag{2.37}$$

and that for $n \geq 4$

$$\int_0^\infty g_n(y)dy = \binom{n-1}{3}. \tag{2.38}$$

We now prove

COROLLARY 2.15. *If $f \in \Lambda^F(m)$ with $\int_0^\infty f(y)dy = 1$ and if m satisfies the hypothesis of Theorem 2.9 (i) and (ii), then for all $n \geq 3$, as $x \to \infty$*

$$\frac{1}{m(x)} \left[1 - G^{xn}(x) - n(1-G(x)) + \binom{n}{2}(1 - 2G(x) + G^{x2}(x)) \right] - 3\mu^2 F \binom{n}{2}$$

Proof. It follows from Corollary 2.11 and Lemma 2.6 that $f^{*i} \in \Lambda^{iF}$. Hence $g_n \in \Lambda^G(m)$ for some constant G. We have to prove that

$$\lim_{x\to\infty} S_n(x)/m(x) = -3\mu^2 F \binom{n}{3}. \tag{2.39}$$

Now we first prove that

$$\lim_{x\to\infty} S_3(x)/m(x) = -3\mu^3 F. \tag{2.40}$$

Define $R_2(x) := 1 - 2G(x) + G^{x2}(x)$ and observe that

$$R_2'(x) = r_2(x), \quad S_3(x) = R_2(x) - R_2 * f(x)$$

In Corollary 2.11 we proved that $r_2(x) \sim 2\mu F m(x)(x \to \infty)$ and hence we have $R_2(x) \in \Lambda^{2\mu F}(m)$. From Theorem 2.9 (ii) it follows that as $x \to \infty$,

$$\frac{1}{m(x)}\left[R_2 * f(x) - R_2(x)\int_0^\infty f(x)dx - f(x)\int_0^\infty R_2(y)dy\right]$$
$$\to 2\mu F\int_0^\infty y f(y)dy + F\int_0^\infty yR_2(y)dy.$$

Using $\int_0^\infty f(y)dy = 1$, $\int_0^\infty R_2(y)dy = 0$ and $\int_0^\infty yR_2(y)dy = \mu^2$ we obtain (2.40). To prove (2.39) use $g \in \Lambda^G(m)$, (2.40) and Theorem 2.9 (i) see that

$$\lim_{x\to\infty}\frac{1}{m(x)}\left[S_3 * g_n(x) - S_3(x)\int_0^\infty g_n(y)dy - g(x)\int_0^\infty S_3(y)dy\right] = G\int_0^\infty yS_3(y)dy.$$

Using (2.36)–(2.38) and (2.40) we obtain (2.39).■

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