PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 43 (57), 1988, 59–63

A NOTE ON CERTAIN CLASS DEFINED BY RUSCHEWEYH DERIVATIVES

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Abstract. The object of this paper is to prove new some results about the class $M(n, \alpha)$ of analytic functions $f(z)$ in the unit disk, defined by Ruscheweyh derivatives $D^n f(z)$. That is, a property of the class $M(n, \alpha)$ and the subordination theorems for Ruscheweyh derivatives $D^n f(z)$ are shown.

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Introduction. Let A denote the class of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

which are analytic in the unit disk $U = \{z :| z |< 1\}$. Let the functions

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (j = 1, 2)$$

be in the class A; then we define the convolution product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n.$$

With the aid of the above convolution product, Ruscheweyh [**7**] has introduced a derivative $D^n f(z)$ of $f(z)$ by

$$D^n f(z) = z(1 - z)^{-(n+1)} * f(z) \quad (n \in N_0 = \{0, 1, 2, \cdots\})$$

for $f(z) \in A$. Note that

$$D^n f(z) = z(z^{n-1} f(z))^{(n)}/n! \qquad (n \in N_0).$$

AMS Subject Classification (1980): Primary 30 C 45

By using the Ruscheweyh derivative $D^n f(z)$, Goel and Sohi [**2**] introduced a subclass $M(n,\alpha)$ of A consisting of functions $f(z)$ which satisfy the condition

$$Re\{D^{n+1}f(z)/z\} > \alpha \qquad (n \in N_0)$$

for some $\alpha(0 \le \alpha < 1)$, and for all $z \in U$. We observe that the class $M(0,\alpha)$ when $n = 0$ is the subclass of A consisting of functions $f(z)$ satisfying the condition $Re\{f'(z)\} > \alpha$ for some $\alpha(0 \le \alpha < 1)$, for all $z \in U$.

Let $f(z)$ and $g(z)$ be analytic un the unit disk U. Then a function $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in the unit disk U satisfying $w(0) = 0$ and $\mid w(z) \mid < 1$ $(z \in U)$ such that $f(x) = g(w(z))$. We denote by $f(z) \prec g(z)$ this relation. If $g(z)$ is univalent in U, then the subordination $f(z) \prec g(z)$ iz equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

The concept of subordination can be traced back to Lindelöf [**3**], but Littlewood [**4**] and Rogosinski [**6**] have introduced the term and discovered the basic relatons.

2. A property of the class $M(n,\alpha)$. Let us recall the following lemma by Nehari [**5**]

Lemma 1. *Let the function* $\Phi(z)$ *be analytic in the unit disk* U *such that* $\mid \Phi(x) \mid \le 1$ *for* $z \in U$. *Then*

$$\mid \Phi'(z) \mid \le (1 - \mid \Phi(z) \mid^2)/(1 - \mid z \mid^2) \qquad (z \in U).$$

With the aid of Lemma 1, we derive

Theorem 1 *Let the function* $f(z)$ *defined by* (1) *be in the class* $M(n,\alpha)$ *with* $0 \le \alpha \le 1/2$ *and* $n \in N_0$. *Then, for* $z \in U$, *we have*

$$Re\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\} \ge \frac{(n+2) - 2(n+3)(1-\alpha) \mid z \mid + (n+2)(1-2\alpha) \mid z \mid^2}{(n+2)(1 - \mid z \mid)\{1 - (1-2\alpha) \mid z \mid\}}$$

Proof. Since $f(z) \in M(n,\alpha)$ implies

$$D^{n+1}f(z)/z \prec (1 + (1 - 2(\alpha)z)/(1-z) \qquad (z \in U),$$

there exists an analytic function $w(z)$ in the unit disk U with $w(0) = 0$ and $\mid w(z) \le 1 (z \in U)$ such that

$$D^{n+1}f(z)/z = (1 + (1-2\alpha)w(z))/(1 - w(z)). \tag{2}$$

Applying the Schwarz lemma, (2) can be written as

$$D^{n+1}f(z)/z = (1 + (1-2\alpha)z\Phi(z))/(1 - z\Phi(z)) \qquad (z \in U), \tag{3}$$

where $\Phi(z)$ is analytic in the unit disk U and satisfies $\mid \Phi(z) \mid \le 1$ for $z \in U$. Making the logarithmic differentiations of both sides in (3), and using the identity

$$z(D^{n+1}f(z))' = (n+2)D^{n+2}f(z) - (n+1)D^{n+1}f(z), \tag{4}$$

we obtain

$$\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} = 1 + \frac{2(1-\alpha)\{z^2\Phi'(z) + z\Phi(z)\}}{(1 - z\Phi(z))\{1 + (1-2\alpha)z\Phi(z)\}}, \quad \text{or}$$

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = 1 + \frac{2(1-\alpha)\{z^2\Phi'(z) + z\Phi(z)\}}{(n+2)(1 - z\Phi(z))\{1 + (1-2\alpha)z\Phi(z)\}}, \tag{5}$$

Therefore, from Lemma 1 and (5), it follows that

$$\begin{aligned} Re\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\} &\geq 1 - \frac{2(1-\alpha)\{\mid z^2\Phi'(z)\mid + \mid z\Phi(z)\mid\}}{(n+2)(1 - \mid z\Phi(z)\mid)\{1-(1-2\alpha)\mid z\Phi(z)\mid\}}, \\ &\geq 1 - \frac{2(1-\alpha)\mid z\mid(\mid z\mid + \mid \phi(z)\mid)}{(n+2)(1-\mid z\mid^2)\{1-(1-2\alpha)\mid z\Phi(z)\mid\}} \\ &\geq 1 - \frac{2(1-\alpha)\mid z\mid}{(n+2)(1-\mid z\mid)\{1-(1-2\alpha)\mid z\mid\}} \\ &= \frac{(n+2) - 2(n+3)(1-\alpha)\mid z\mid + (n+2)(1-2\alpha)\mid z\mid^2}{(n+2)(1-\mid z\mid)\{1-(1-2\alpha)\mid z\mid\}} \end{aligned}$$

which completes the assertion of the Theorem.

Taking $n = 0$ in Theorem 1, we have

Corollary 1. *Let the function $f(z)$, defined by* (1), *be in the class $M(0, \alpha)$ for $0 \leq \alpha \leq 1/2$. Then, for $z \in U$, we have*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \frac{1 - 3(1-\alpha)\mid z\mid + (1-2\alpha)\mid z\mid^2}{(1-\mid z\mid)\{1-(1-2\alpha)\mid z\mid\}}.$$

3. Subordination Results. We need the following results by Eenigenburg, Miller, Mocanu and Reade [**1**].

Lemma 2 *Let the function $p(z)$ and $h(z)$ be analytic in the unit disk U such that $p(0) = h(0) = 1$. Further, let $h(z)$ be a convex and univalent function in the unit disk U satisying the condition $Re\{\beta h(z) + \gamma\} > 0$ for complex numbers β, γ and for all $z \in U$. If $p(z), h(z), \beta$ and γ satisfy the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \textit{then } p(z) \prec h(z) \quad (z \in U).$$

Lemma 3. *Under the hypotheses of Lemma* 2, *if the Briot-Bouquet differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \qquad (q(0) = 1)$$

has a univalent solution, then $p(z) \prec q(z) \prec h(z)$. Furthermore, $q(z)$ is the best dominant

Applying the lemmas above, we derive

THEOREM 2. *Let a function $h(z)$ be convex and univalent in the unit disk U such that $h(0) = 1$ and $Re\{h(z)\} > 0$ for $z \in U$. For $f(z)$ belonging to A and $n \in N_0$, if*

$$D^{n+2}f(z)/z \prec h(z), \text{ then } D^{n+1}f(z)/z \prec h(z) \quad (z \in U).$$

Proof. Defining the function $p(z)$ by

$$p(z) = D^{n+1}f(z)/z, \tag{6}$$

we know that $p(z)$ is analytic in the unit disk U with $p(0) = 1$. Differentiating both sides of (6), and applying (4), we have

$$(n+2)D^{n+2}f(z)/z - n(n+1)D^{n+1}f(z)/z = p(z) + zp'(z),$$

that is

$$D^{n+2}f(z)/z = p(z) + zp'(z)/(n+2) \prec h(z).$$

Consequently, by taking $\beta = 0$ and $\gamma = n + 2$ in Lemma 2, we complete the proof of Theorem 2.

Letting $n = 0$ in Theorem 2, we have

COROLLARY 2. *Under the hypothesis in Theorem 2,*

if

$$f'(z)xf''(z)/2 \prec h(z), \text{ then } f'(z) \prec h(z) \quad (z \in U).$$

Further, by putting $h(z) = \{1 + (1 - 2\alpha)z\}/(1 - z)$ in Theorem 2, we have

COROLLARY 3. [**2**] *For $0 \leq \alpha \leq 1$ and $n \in N_0$, we have $M(n+1, \alpha) \subset M(n, \alpha)$.*

Finally, we prove

THEOREM 3. *Under the hypotheses of Theorem 2, if the Briot-Bouquet differential equation*

$$q(z) + zq'(z)/(n+2) = h(z) \qquad (q(0) = 1)$$

has a univalent solution, then

$$D^{n+1}f(z)/z \prec q(z) \prec h(z) \tag{7}$$

Furthermore, $q(z)$ is the best dominant.

Proof. If we replace $p(z$ by $D^{n+1}f(z)/z$ and take $\beta = 0$ and $\gamma = n + 2$ in Lemmas 2 and 3, we see that the result follows from (7).

Acknowledgement. This research was completed at the Department of Mathematics, Faculty of Tehnology and Metallurgy, Belgrade, Yugoslavia while the second author was on study leave from Kinki University, Higashi-Osaka, Osaka 577, Japan.

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(Received 26 06 1986)