PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 43 (57), 1988, 65–69

DISTORTION THEOREMS FOR FRACTIONAL CALCULUS OF CERTAIN ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

Sang Keun Lee, Shigeyoshi Owa, Tadayuki Sekine and Milutin Obradović

Abstract. We give some distortion theorems for fractional calculus of analytic functions with negative coefficients belonging to a certain generalized class $T_k(j,\alpha)$ introduced by Owa and Lee [**5**].

.

Introduction. Let T_k be the class of functions of the form

$$f(z) = z - \sum_{n=k+1}^{\infty} a_n z^n \qquad (a_n \geq 0; k \in N = \{1,2,3,\dots\}) \tag{1}$$

which are analytic in the unit disk $U = \{z :| z |< 1\}$.

For $f(z)$ in T_k, we define

$D^0 f(z) = f(z), D^1 f(z) = Df(z) = zf'(z),$

$D^j f(z) = D(D^{j-1} f(z)) \qquad (j \in N).$

The above differential operator D^j was introduced by Salagean [**8**].

With the differential operator D^j, a function $f(z)$ in T_k is said to be in the class $T_k(j,\alpha)$ if and only if

$$Re\{D^j f(z)/z\} > \alpha \qquad (j \in N \cup \{0\})$$

for some $\alpha (0 \leq \alpha < 1)$, and for all $z \in U$.

In order to show our distortion theorems for fractional calculus of functions in $T_k(j,\alpha)$, we need the following lemma due to Owa and Lee [**5**].

Lemma. *Let the function $f(z)$ be in the class T_k. Then $f(z)$ is in the class $T_k(j,\alpha)$ if and only if $\sum_{n=k+1}^{\infty} n^j a_n \leq 1 - \alpha$.*

AMS Subject Classification (1980): Primary 30C45

2. Distortion theorems for fractional calculus. Many essentially equivalent definitions of the fractional calculus, that is the fractional derivatives and the fractional integrals, can be found in the literature ([**1**,**2**, **6**] and [**7**]. We find it convenient to recall here the following definitions which were used recently by Owa ([**3**, **4**]).

Definition 1. The fractional integral of order λ is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi,$$

where $\lambda > 0, f(z)$ is an analytic function in a simply conneted region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi)$

Definition 2. The fractional derivative of order λ is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\Lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi,$$

where $0 \leq \lambda < 1, f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda} f(z) = d^n D_z^{\lambda}(z)/dz^n \qquad (0 \leq \lambda < 1;\ n \in N \cup \{0\}).$$

Now, we prove

THEOREM 1. *Let the function $f(z)$, defined by* (1), *be in the class $T_k(j,\alpha)$. Then*

$$| D_z^{-\lambda}(D^i f(z)) | \geq \frac{| z |^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{\Gamma(k+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(k+2+\lambda)(k+1)^{j-1}} | z |^k \right\}, \tag{2}$$

$$| D_z^{-\lambda}(D^i f(z)) | \leq \frac{| z |^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 + \frac{\Gamma(k+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(k+2+\lambda)(k+1)^{j-1}} | z |^k \right\} \tag{3}$$

For $\lambda > 0,\ 0 \leq i \leq j$, and $z \in U$. The equalities in (2) *and* (3) *are attained for the function $f(z)$ given by*

$$f(z) = z - (1-\alpha)(k+1)^{-j} z^{k+1}. \tag{4}$$

Proof. We note that

$$\Gamma(2+\lambda) z^{-\lambda} D_z^{-\lambda}(D^i f(z)) = z - \sum_{n=k+1}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} n^i a_n z^n \tag{5}$$

Defining the function $\varphi(n)$ by

$$\varphi(n)=\Gamma(n+1)\Gamma(2+\lambda)/\Gamma(n+1+\lambda) \qquad (n\geq k+1),$$

we can see that $\varphi(n)$ is decreasing in n, that is, that

$$0<\varphi(n)\leq\varphi(k+1)=\Gamma(k+2)\Gamma(2+\lambda)/\Gamma(k+2+\lambda). \tag{6}$$

On the other hand, our Lemma implies

$$\sum_{n=k+1}^{\infty} n^i a_n \leq (1-\alpha)(k+1)^{-(j-1)} \qquad (0\leq i\leq j). \tag{7}$$

Therefore, by using (5), (6) and (7), we have

$$\begin{aligned} \mid \Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}(D^i f(z)) &\geq \mid z\mid -\varphi(k+1)\mid z\mid^{k+1}\sum_{n=k+1}^{\infty} n^i a_n \\ &\geq -\frac{\Gamma(k+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(k+2+\lambda)(k+1)^{j-1}}\mid z\mid^{k+1} \end{aligned}$$

which gives (2), and

$$\begin{aligned} \mid \Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}(D^i f(z)) &\leq \mid z\mid +\varphi(k+1)\mid z\mid^{k+1}\sum_{n=k+1}^{\infty} n^i a_n \\ &\leq \mid z\mid +\frac{\Gamma(k+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(k+2+\lambda)(k+1)^{j-1}}\mid z\mid^{k+1} \end{aligned}$$

which shows (3)

Further, since the equalities in (2) and (3) are attained for the function $f(z)$ defined by

$$D_z^{-\lambda}(D^i f(z))=\frac{z^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{\Gamma(k+2)\Gamma(2+\lambda)(1-\lambda)}{\Gamma(k+2+\lambda)(k+1)^{j-1}}z^k\right\},$$

that is, defined by (4), the proof of Theorem 1 is copmleted.

Taking $i=0$ in Theorem 1, we have:

Corollary 1. *Let the function $f(z)$ defined by* (1) *be in the class $T_k(j,\alpha)$. Then*

$$\mid D_z^{-\lambda}f(z)\mid \geq \frac{\mid z\mid^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{\Gamma(k+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(k+2+\lambda)(k+1)^{j}}\mid z\mid^k\right\}, \tag{8}$$

$$\mid D_z^{-\lambda}f(z)\mid \leq \frac{\mid z\mid^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1+\frac{\Gamma(k+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(k+2+\lambda)(k+1)^{j}}\mid z\mid^k\right\} \tag{9}$$

for $\lambda>0$ and $z\in U$. The equalities in (8) *and* (9) *are attained for the function $f(z)$ given by* (4).

Remark. Letting $\lambda \to 0$ in Corollary, we have the former result by Owa and Lee [**5**].

Next, we prove

THEOREM 2. *Let the function $f(z)$ defined by* (1) *be in the class $T_k(j,\alpha)$. Then*

$$| D_z^\lambda(D^i f(z)) | \geq \frac{| z |^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 - \frac{\Gamma(k+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(k+2+\lambda)(k+1)^{j-1}} | z |^k \right\}, \tag{10}$$

$$| D_z^\lambda(D^i f(z)) | \leq \frac{| z |^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 + \frac{\Gamma(k+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(k+2+\lambda)(k+1)^{j-1}} | z |^k \right\} \tag{11}$$

for $0 \leq \lambda < 1$, $0 \leq i \leq j-1$, and $z \in U$. The equalities in (10) *and* (11) *are attained for the function $f(z)$ given by* (4).

Proof. It is easy to see that

$$\Gamma(2-\lambda) z^\lambda D_z^\lambda (D^i(z)) = z - \sum_{n=k+1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} n^i a_n z^n. \tag{12}$$

Since the function

$$\psi(n) = \Gamma(n)\Gamma(2-\lambda)/\Gamma(n+1-\lambda) \qquad (n \geq k+1)$$

is decreasing in n, we have

$$0 < \psi(n) \leq \psi(k+1) = \Gamma(k+1)\Gamma(2-\lambda)/\Gamma(k+2-\lambda). \tag{13}$$

Further, note that our Lemma gives

$$\sum_{n=k+1}^{\infty} n^{i+1} a_n \leq (1-\alpha)(k+1)^{-(j-i-1)} \tag{14}$$

for $f(z) \in T_k(j,\alpha)$. It follows from (12), (13), and (14) that

$$\begin{aligned} | \Gamma(2-\lambda) z^\lambda D_z^\lambda (D^i f(z)) | &\geq | z | - \psi(k+1) | z |^{k+1} \sum_{n=k+1}^{\infty} n^{i+1} a_n \\ &\geq | z | - \frac{\Gamma(k+2)\Gamma(2-\lambda)(1-\alpha)}{\Gamma(k+2-\lambda)(k+1)^{j-1}} | z |^{k+1} \end{aligned}$$

which implies (10), and that

$$\begin{aligned} | \Gamma(2-\lambda) z^\lambda D_z^\lambda (D^i f(z)) | &\leq | z | + \psi(k+1) | z |^{k+1} + \sum_{n=k+1}^{\infty} n^{i+1} a_n \\ &\leq | z | + \frac{\Gamma(k+2)\Gamma(2-\lambda)(1-\alpha)}{\Gamma(k+2-\lambda)(k+1)^{j-1}} | z |^{k+1} \end{aligned}$$

which gives (11).

Finally, we can see that the equalities in (10) and (11) are attained for the function $f(z)$ defined by

$$D_z^\lambda(D^i f(z)) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 - \frac{\Gamma(k+2)\Gamma(2-\lambda)(1-\alpha)}{\Gamma(k+2-\lambda)(k+1)^{j-1}} z^k\right\}.$$

This completes the proof of Theorem 2.

Making $i = 0$ in Theorem 2, we have

COROLLARY 2. *Let the function $f(z)$ defined by* (1) *be in the class $T_k(j,\alpha)$.* Then

$$| D_z^\lambda f(z) | \geq \frac{| z |^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 - \frac{\Gamma(k+2)\Gamma(2-\lambda)(1-\alpha)}{\Gamma(k+2-\lambda)(k+1)^j} | z |^k\right\}, \tag{15}$$

$$| D_z^\lambda f(z) | \leq \frac{| z |^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 + \frac{\Gamma(k+2)\Gamma(2-\lambda)(1-\alpha)}{\Gamma(k+2-\lambda)(k+1)^j} | z |^k\right\} \tag{16}$$

for $0 \leq \lambda < 1$ *and* $z \in U$. *The equalities in* (15) *and* (16) *are attained for the function* $f(z)$ *given by* (4).

Remark 2. Letting $\lambda = 0$ or $\lambda \to 1$ in Corollary 2, we have the former theorems due to Owa and Lee [**5**].

REFERENCES

[1] K. Nichimoto, *Fractional derivative and integral. Part I* , j. Coll. Engrg. Nihon Univ. **B-17** (1976), 11–19.

[2] T. J. Osler, *Leibniz rule for fractional derivative generalized and an application to infinite series*, SIAM J. Appl. Math. **18** (1970), 658–674.

[3] S. Owa, *On the distortion theorems, I*, Kyungpook Math. **18** (1978), 53–59.

[4] S. Owad, *Some applications of the fractional calculus*, Research Notes in Math. 138, Pitman, Boston, London and Melbourne, 1985, 164–175.

[5] S. Owa and S. K. Lee, *Certain generalized class of analytic functions with negative coefficients*, to appear.

[6] B. Ross, *A Brief History and Exposition of the Fundamental Theory of Fractional Calculus*, Lecture Notes in Math. 457, Springer-Verlag, Berlin Heidelberg and New York, 1975, 1–36.

[7] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Coll. General Ed. Kyushu Univ. **11** (1978), 135–143.

[8] G. S. Salagean, *Subclasses of Univalent Functions*, Lecture Notes in Math. 1013, Springer-Verlag, Berlin, Heidelberg and new York, 1983, 362–372.

S. K. Lee
Department of Mathematics
Gyeongsang National University
Jinju 620
Korea

S. Owa
Department of Mathematics
KinkiUniversity
Higashi-Osaka, Osaka 577
Japan

T. Sekine:
Department of Mathematics
College of Science
and Tehnology
Nihon University
Chiyoda, Tokyo 101
Japan

M. Obradović:
Katedra za matematiku
Tehnološko-metalurški Fakultet
Karnegijeva 4
11000 Beograd
Yugoslavia

(Received 12 11 1986)