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# A NOTE ON LENGTH DISTORTION FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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**Abstract**. Some interesting results on length distortion for certain classes of analytic functions have been proved recently by Y. Komatu. We derive some results involving analytic functions satisfying $Re\{f(z)/z\} > \alpha$, starlike functions of order $\alpha$, and convex functions of order $\alpha$.

**Introduction**. Let $A$ denote the class of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

which are analytic in the unit disk $U = \{z : | z |< 1\}$. Komatu [**1**] introduced a linear operator $L$ of the form

$$Lf(z) = \int_0^1 t^{-1} f(zt) d\sigma(t)$$

defined on $A$ where $\sigma(t)$ denotes a probability measure supported by the interval $I = [0, 1]$. He has shown in [**1**] that there exiats a family of operators $\{L^\lambda\}$ depending on a continuous parametar $\lambda$ such that

$$L^\lambda L^\mu = L^{\lambda+\mu} \qquad \text{where } L^0 = \mathrm{id}$$

and, under certain conditions on $\sigma$,

$$f_\lambda(z) \equiv L^\lambda f(z) = \int_I t^{-1} f(zt) d\sigma_\lambda(t) \tag{1.1}$$

where $\sigma_\lambda$ is a probability measure supported by $I$.

Let $R(\alpha) \subset A$ be the class of all functions satisfying the condition

$$Re\{f(z)/z\} > \alpha \qquad (f(z) \in A)$$

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*AMS Subject Classification* (1980): Primary 30C45 for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in U$.

The main tool in deriving our results for length distortion is based on the following lemma [**2**].

LEMMA 1. *Let* $L_\lambda(r)(\lambda \geq 0)$ *be the length of the image-curve of* $\{z : | z |= r < 1\}$ *by a mapping* $w = f_\lambda(z)/z$ *caused by* (1.1) *with any* $f(z) \in R(0)$. *Then we have*

$$L_\lambda(r) \leq \int_I L(rt) d\sigma_\lambda(t) \tag{1.2}$$

*where* $L = L_0$ *satisfies* $L(r) \leq 4\pi r/(1 - r^2)$.

**2. The class R**$(\alpha)$. In view of Lemma 1, we state and prove

THEOREM 1. *Let* $L_\lambda(r)(\lambda \geq 0)$ *be the length of the image-curve of* $\{z : | z |= r < 1\}$ *by a mapping* $w = f_\lambda(z)/z$ *caused by* (1.1) *with any* $f(z) \in R(\alpha)$. *Then we have the estimation* (1.2) *where* $L = L_0$ *satisfies*

$$L(r) \leq 4\pi(1 - \alpha)r/(1 - r^2). \tag{2.1}$$

*Proof*. The estimation (1.2) is obtained by the method of proof in [**2**]. For a function $f(z)$ belonging to the class $R(\alpha)$, the Herglotz representation implies

$$\frac{f(z)}{z} - \alpha = (1 - \alpha) \int_{-\pi}^{\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\tau(\varphi) \tag{2.2}$$

with a probability measure $\tau$ supported by $(-\pi, \pi]$. It follows from (2.2) that

$$g(z) \equiv \frac{f(z)}{z} = \int_{-\pi}^{\pi} \frac{e^{i\varphi} + (1 - 2\alpha)z}{e^{i\varphi} - z} d\tau(\varphi).$$

Therefore, we obtain

$$\begin{aligned} L(r) &= r \int_{-\pi}^{\pi} | g'(re^{i\theta}) | d\theta \\ &\leq 2(1 - \alpha)r \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} | e^{i\varphi} - re^{i\theta} |^{-2} d\tau(\varphi \\ &= 2(1 - \alpha)r \int_{-\pi}^{\pi} d\tau\varphi \int_{-\pi}^{\pi} | e^{i\varphi} - re^{i\theta} |^{-2} d\theta \\ &= 2(1 - \alpha)r \int_{-\pi}^{\pi} 2\pi(1 - r^2)^{-1} d\tau(\varphi) \\ &= 4\pi(1 - \alpha)r/(1 - r^2) \end{aligned}$$

whish gives (2.1). Furthermore, the bound in (2.1) is attained by the extremal function given by $f(z) = (z + (1 - 2\alpha)\varepsilon z)/(1 - \varepsilon z) \qquad (| \varepsilon |= 1)$.

*Remark* 1. By letting $\alpha = 1/2$ in Theorem 1, we have the corresponding result due to Komtu [**2**, Theorem 2].

**3. The class** $S^*(\alpha)$. A function $f(z)$ belonging to the class $A$ is said to be starlike of order $\alpha(0 \le \alpha < 1)$ if it satisfies $Re(zf'(z)/f(z)\} > \alpha$ for some $\alpha(0 \le \alpha < 1)$ and for all $z \in U$. We denote by $S^*(\alpha)$ the class of all functions in $A$ which are starlike of order $\alpha$. We note that $S^*(\alpha) \subseteq S^*(0) \equiv S^*$ for $0 \le \alpha < 1$.

For the function $f(z)$ in the class $S^*(\alpha)$, we prove

THEOREM 2. *Let $L_\lambda(r)(\lambda \ge 0)$ be the length of the image-curve of $\{z : \mid z \mid = r < 1\}$ by a mapping $w = f_\lambda(z)/z$ caused by* (1.1) *with any $f(z) \in S^*(\alpha)$. Then we have the estimation* (1.2) *where $L = L_0$ satisfies*

$$L(r) \le \frac{8(1-\alpha)r}{(1-r)^{2(1-\alpha)}(1+r)} K\left(\frac{2\sqrt{r}}{1+r}\right),$$

*where $K$ denotes the complete elliptic integral of the first kind with modulus $2\sqrt{r}/(1+r)$.*

*Proof*. Again we use the estimation (1.2). Applying Herglotz representation, we have, for $g(z) = f(z)/z$ and $f(z) \in S^*(\alpha)$,

$$\begin{aligned} g'(z) =& f(z)z^{-2}\{(zf'(z)/f(z) - \alpha) - (1-\alpha)\} \\ =& \frac{g(z)}{z}\left\{(1-\alpha)\int_{-\pi}^{\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\tau(\varphi) - (1-\alpha)\right\} \\ =& 2(1-\alpha)g(z)\int_{-\pi}^{\pi} (e^{i\varphi}-z)^{-1} d\tau(\varphi) \end{aligned} \tag{3.1}$$

with a probability measure $\tau$ supported by $(-\pi, \pi]$, Thus from (3.1) we derive

$$\begin{aligned} L(r) =& r\int_{-\pi}^{\pi} \mid g'(re^{i\theta}) \mid d\theta \\ =& 2(1-\alpha)r\int_{-\pi}^{\pi} \mid g(re^{i\theta}) \mid d\theta \mid \int_{-\pi}^{\pi} (e^{i\varphi} - re^{i\theta})^{-1} d\tau(\varphi) \mid \\ \le& 2(1-\alpha)r\int_{-\pi}^{\pi} d\tau(\varphi)\int_{-\pi}^{\pi} \left| \frac{g(re^{i\theta})}{e^{i\varphi} - re^{i\theta}} \right| d\theta \end{aligned}$$

We note that a function $f(z)$ belonging to the class $S^*(\alpha)$ satisfies [**4**]:

$$r(1+r)^{-2(1-\alpha)} \le \mid (re^{i\theta}) \mid \le r(1-r)^{-2(1-\alpha)}$$

which implies

$$\mid g(re^{i\theta}) \mid \le (1-r)^{-2(1-\alpha)}, \tag{3.2}$$

and we have the following explicit expression [**2**]:

$$J(r) = 2\int_0^{\pi} (1 + r^2 - 2r\cos\theta)^{-1/2} d\theta = \frac{4}{1+r} K\left(\frac{2\sqrt{r}}{1+r}\right). \tag{3.3}$$

Consequently, by using (3.2) and (3.3), we obtain

$$L(r) \leq \frac{2(1-\alpha)r}{(1-r)^{2(1-\alpha)}} \int_{-\pi}^{\pi} \frac{1}{\mid 1-re^{i\theta} \mid} d\theta = \frac{2(1-\alpha)r}{(1-r)^{2(1-\alpha)}} K\left(\frac{2\sqrt{r}}{1+r}\right).$$

This completes the proof of Theorem 2.

*Remark* 2. For $\alpha = 0$, Theorem 2 implies the result for $f(z) \in S^*$ proved in [**2**].

**4. The class** $K(\alpha)$. A function $f(z) \in A$ is said to be convex of order $\alpha(0 \leq \alpha < 1)$ if it satisfies $Re\{1 + zf''(z)/f'(z)\} > \alpha$ for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in U$. We denote by $K(\alpha)$ the class of all functions in $A$ which are convex of order $\alpha$. Note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, and that $K(\alpha) \subseteq K(0) \equiv K$ for $0 \leq \alpha < 1$.

In order to derive our theorem for $f(z) \in K(\alpha)$, we recall here the following lemma by MacGregor [**3**].

LEMMA 2. *If $f(z) \in K(\alpha)$, then $f(z) \in S^*(\beta(\alpha))$, where $\beta(\alpha)$ is given by*

$$\beta(\alpha) = \begin{cases} (2\alpha - 1)/2(1 - 2^{1-2\alpha}) & (\alpha \neq 1/2) \\ 1/2 \log 2 & (\alpha = 1/2). \end{cases}$$

Commbining Theorem 2 and Lemma 2, we have

THEOREM 3. *Let $L_\lambda(r)$ $(\lambda \geq 0)$ be the length of the image-curve of $\{z : \mid z \mid = r < 1\}$ by mapping $w = f_\lambda(z)/z$ caused by* (1.1) *with any $f(z) \in K(\alpha)$. Then we have the estimation* (1.2) *where $L = L_0$ satisfies*

$$L(r) \leq \frac{8(1-\beta(\alpha))r}{(1-r)^{2(1-\beta(\alpha))}(1+r)} K\left(\frac{2\sqrt{r}}{1+r}\right).$$

*where $K$ denotes the complete elliptic integral of the first kind with modulas* $2\sqrt{r}/(1+r)$.

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