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ON A SYSTEM OF FUNCTIONAL EQUATIONS

Jovan D. Kečkić

Dedicated to Professor D. S. Mitrinović on the occasion of his eightieth birthday.

Abstract. We consider the system of functional equations (3), written in the matrix form (4). The general solution of this system is obtained provided that there exist $\lambda, \mu \in \mathbf{C}$ such that $\lambda A + \mu B$ is nonsingular. The solutions are expressed by means of generalized inverses of matrices.

1. Let S be a nonempty set, and suppose that $g : S \to S$ is such that $g^n x = x$ for all $x \in S$ and some fixed positive integer n. The functional equation

$$af(gx) = bf(x), \tag{1}$$

where a, b are given complex numbers and the unknown function f maps S into the set of all complex numbers $\mathbf{C}$, is well known. It is, of course, a special case of the linear cyclic equation (see, for example, [**1**]), but it was also considered independently, e. g. by Mitrinović [**2, 3**]. If $a^n \neq n^n$, the general solution of (1) is trivial, $f(x) \equiv 0$; if $a^n = b^n$, its general solution is given by

$$f(x) = \sum_{k=0}^{n-1} a^k b^{n-k-1} h(g^k x) \quad (h : S \to \mathbf{C} \text{ arbitrary}). \tag{2}$$

In this note we shall be concerned with the system of functional equations

$$\sum_{j=1}^{m} a_{ij} f_j(gx) = \sum_{j=1}^{m} b_{ij} f_j(x) \qquad (i = 1, 2, \dots, m), \tag{3}$$

where g is the same as before, $a_{ij} \in \mathbf{C},\ b_{ij} \in \mathbf{C} (i, j = 1, \dots, m)$ are given, and the unknown functions $f_1, \dots, f_m$ map S into $\mathbf{C}$. The matrix form of (3), namely

$$AF(gx) = BF(x) \tag{4}$$

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where $A = \| a_{ij} \|_{m\times m}, B = \| b_{ij} \|_{m\times m}, F(x) = \| f_1(x) \cdots f_m(x) \|^T$, is completely analogous to (1), but as we shall see its general solution cannot be obtained so simply. In fact, we shall be needing some specijal generalized inverses of matrices.

Recall that for any complex matix M there exists an infinity of matrices X such that $MXM = M$. Any such matrix X is called a (1)-inverse of M and is denoted by $M^{(1)}$. Again, for any square complex matix M there exists a unique matrix X such that $MX = XM, XMX = X, M^{s+1}X = M^s$, where s is the index of M, i. e. the smallest nonnegative integer such that rank $M^s =$ rank M^{s+1}. This matrix X is called the Drazin inverse of M and is denoted by M_D. Many properties of those two, and other generalized inverses, can be found for example in [**4**].

2. We shall first solve the equation (4) on the condition that one of the coefficients A, B is the unit matrix.

THEOREM 1. *The general solution of the equation*

$$AF(gx) = F(x) \tag{5}$$

is given by

$$F(x) = \sum_{m=0}^{n-1} A^k (I - C^{(1)}C) H(g^k x). \tag{6}$$

where $C =: A^n - I$, $H(x) = \| h_1(x) \dots h_m(x) \|^T$, *and* $h_1, \dots, h_m$ *are arbitrary functions which map* S *into* $\mathbf{C}$.

Proof. From (6) we get

$$\begin{aligned} AF(gx) =& \sum_{k=0}^{n-1} A^{k+1}(I - C^{(1)}C)H(g^{k+1}x) \\ =& A^n (I - C^{(1)}C)H(x) + \sum_{k=1}^{n-1} A^k (I - C^{(1)}C)H(g^k x). \end{aligned}$$

But $A^n(I - C^{(1)}C) = (I + C)(I - C^{(1)}C) = I - C^{(1)}C$, and so

$$AF(gx) = \sum_{k=1}^{n-1} A^k (I - C^{(1)}C)H(g^k x) = F(x),$$

which means thay (6) is a solution of (5).

Conversely, suppose that $F_0(x)$ is a solution of (5), i.e. that $AF_0(gx) = F_0(x)$. But then $A^k F_0(g^k x) = F_0(x)$ for all nonnegative integers k; in particular for $k = n$ we get $A^n F_0(x) = F_0(x)$, i.e. $CF_0(x) = 0$ implying $CF_0(g^k x) = 0$ for $k = 0, 1, \dots, n-1$. Put $H(x) = F_0(x)/n$ into (6). We get

$$F(x) = \frac{1}{n}\sum_{k=1}^{n-1} A^k (I - C^{(1)}C)F_0(g^k x) = \frac{1}{n}\sum_{k=1}^{n-1} A^k F_0(g^k x) = \frac{1}{n}\sum_{k=1}^{n-1} F_0(x) = F_0(x).$$

This means that the solution $F_0(x)$ can be obtained from (6), i.e. that (6) is general solution of (5).

In a similar manner the following theorem is proved.

THEOREM 2. *The general solution of the equation*

$$F(gx) = BF(x) \tag{7}$$

is given by

$$F(x) = \sum_{k=1}^{n-1} B^{n-k-1}(I - D^{(-1)}D)H(g^k x),$$

where $D =: B^n - I$ *and* $H(x)$ *is the same as in Theorem 1.*

Remark 1. Equatons of the form

$$AF(g^p x) = F(g^q x), \quad F(g^p x) = BF(g^q x) \qquad (1 < p,\ q < n)$$

are easoly reduced to the forms (5) or (7).

Remark 2. There is some analogy between the general solution of the matrix equation (5) and the general solution of the scalar equation (1) for the case $b = 1$. Namely, if $A^n - I$ is nonsingular, then $C^{(1)} = C^{-1}$, and so $I - C^{(1)}C = 0$, implying that $F(x) \equiv 0$ is the only solution of (5). Again, if $A^n - I = 0$, then $C = 0$, and (6) reduces to the form complately analogous to (2) with $b = 1$. Of course, the case when $A^n - I$ is nonzero, but singular, has no analogy in the scalar case.

3. We now turn to the general equation (4). If either one of the coefficients A, B is nonsingular, then (4) can be reduced to one of the forms (5), (7); hence, the only interesting case in when both A and B are singular. Using the known general solution of (1) and Theorem 1 and 2, one is tempted to try

$$F(x) = \sum_{k=0}^{n-1} A^k B^{n-k-1}(I - E^{(1)}E)H(g^k x) \tag{9}$$

for a solution, where $E =: A^n - B^n$, and $H(x)$ is the same as in Theorem 1. Indeed, if we introduce a rather restricting supposition that $AB = BA$, then it is easily verifed that (9) is a solution of (4), but unfortunately this solution need not be general. More precisely, if E is nonsingular, then (9) does give the general solution of (4), that is the trivial solution. If E is singular (zero or not) then (9) need not be the general solution of (4), as shown by the following two examples.

Example 1. Let $m = n = 2,\ A = 0,\ B = \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix}$. The equation (4) reduces to $f_1(x) + f_2(x) = 0$, and clearly has nontrivial solutions. On the other hand, (9) becomes $F(x) = B(I - E^{(1)}E)H(gx)$, with $E = -B^2$, and so we may take $E^{(1)} = -(1/4)I$. But then it is easily verified that $B(I - E^{(1)}E) = 0$, and so (9) gives only the trivial solution.

Example 2. Let $m = n = 2, A = B$, where B is the same as in Example 1. The equation (4) reduces to the equation $f_1(gx)+f_2(gx) = f_1(x)+f_2(x)$, with the general solution

$$f_1(x) = p(x), \quad f_2(x) = q(x) + q(gx) - p(x), \tag{10}$$

where p and q are arbitrary. On the other hand, $E = A^2 - B^2 = 0$ and formula (9) becomes $F(x) = A(H(x) + H(gx))$, i. e. $f_1(x) = f_2(x) = \varphi(x) + \varphi(gx)$, where φ is arbitrary, and this solution does not contain all the solutions (10).

Hence, the attempt to exploit the analogy with the corresponding scalar equation leads to two pretty weak conclusions: (i) If $AB = BA$ and if $A^n - B^n$ is nonsingular, then the general solution of (4) is $F(x) \equiv 0$; (ii) If $AB = BA$, then (9) is a solution (not necessarily general) of (4).

4. Systems of diferential and difference equations of the form

$$AF'(x) = BF(x), \qquad AX_{n+1} = BX_n \tag{11}$$

were successfully solved in [**4**] on the condition that $\lambda A - B$ is nonsingular for some $\lambda \in \mathbf{C}$. If such a λ exists, the equatons (11) are called tractable, for reasons which cannot be directly carried over to the equation (4).

We introduce now the same condition, i. e. we suppose that there exists a $\lambda \in \mathbf{C}$ such that $\lambda A - B$ is nonsingular.

Multiply (4) by $(\lambda A - B)^{-1}$ to obtain

$$PF(gx) = QF(x) \tag{12}$$

with

$$P = (\lambda A - B)^{-1}A, \quad Q = (\lambda A - B)^{-1}B. \tag{13}$$

From (13) we get $\lambda P - Q = I$, implying that $Q = \lambda P - I$, and that $PQ = QP$, so that (12) becomes

$$PF(gx) = (\lambda P - I)F(x). \tag{14}$$

Since A is, by hypothesis, singular, so is P, and hence there exist nonsingular matrices S and R, and a nilpotent matrix N such that

$$P = S \begin{Vmatrix} N & O \\ O & R \end{Vmatrix} S^{-1}. \tag{15}$$

If we put $F(x) = S \parallel G(x)\, K(x) \parallel^T$, the equation (14) can be rewritten as

$$\begin{Vmatrix} N & O \\ O & R \end{Vmatrix} \begin{Vmatrix} G(gx) \\ K(gx) \end{Vmatrix} = \begin{Vmatrix} \lambda N - I & 0 \\ 0 & \lambda R - I \end{Vmatrix} \begin{Vmatrix} G(x) \\ K(x) \end{Vmatrix},$$

and it splits into

$$NG(gx) = (\lambda N - I)G(x), \tag{16}$$

$$RK(gx) = (\lambda R - I)K(x). \tag{17}$$

Since N is nilpotent, $\lambda N - I$ is nonsingular and from (16) we get

$$\begin{aligned} G(x) &= (\lambda N - I)^{-1} N G(gx) = (\lambda N - I)^{-2} N^2 G(g^2 x) \\ &= \cdots = (\lambda N - I)^{-s} N^s G(g^s x) = 0, \end{aligned}$$

where s is the index of nilpotency of N.

On the other hand, R is nonsingular, and hence (17) is equivalent to

$$K(gx) = WK(x), \qquad W = R^{-1}(\lambda R - I).$$

This is an equation of the form (7) and its general solution is therefore

$$K(x) = \sum_{k=0}^{n-1} W^{n-k-1}(I - Y^{(1)}Y)Z(g^k x), \qquad Y = W^n - I,$$

where $Z(x)$ is arbitrary, of the same size as $K(x)$.

Hence, the general solution of (12) is given by

$$F(x) = S \begin{Vmatrix} 0 \\ \sum_{k=0}^{n-1} W^{n-k-1}(I - Y^{(1)}Y)Z(g^k x) \end{Vmatrix}. \tag{18}$$

The general solution (18) of (12) can be written in a neather form by using the Drazin inverse of P. Namely, recall that if P is given by (15), then the Drazin inverse P_D of P is

$$P_D = S \begin{Vmatrix} 0 & 0 \\ 0 & R^{-1} \end{Vmatrix} S^{-1}.$$

An alternative expression for the general solution of (12) is given by the following theorem.

THEOREM 3. *Suppose that there exists a $\lambda \in \mathbf{C}$ such that $\lambda A - B$ is nonsingular. The general solution of the equation (4) is given by*

$$F(x) = \sum_{k=0}^{n-1} (P_D Q)^{n-k-1} P_D P (I - M^{(1)}M) H(g^k x), \tag{19}$$

where $P = (\lambda A - B)^{-1}A$, $Q = (\lambda A - B)^{-1}B$, $M = (P_D Q)^n - P_D P$ and H is the same as in Theorem 1.

Proof. The general solution of (12) is (18), and the equatons (4) and (12) are equivalent. We shall therefore only prove that expressions $F(x)$ given by (18) and (19) are the same.

First, we note that from

$$\begin{aligned} M &= (P_D Q)^n - P_D P = S \begin{Vmatrix} 0 & 0 \\ 0 & R^{-1} \end{Vmatrix} S^{-1} S \begin{Vmatrix} (\lambda N - I)^n & 0 \\ 0 & (\lambda R - I)^n \end{Vmatrix} S^{-1} \\ &- S \begin{Vmatrix} 0 & 0 \\ 0 & R^{-1} \end{Vmatrix} S^{-1} S \begin{Vmatrix} N & 0 \\ 0 & R \end{Vmatrix} S^{-1} = S \begin{Vmatrix} 0 & 0 \\ 0 & W^n \end{Vmatrix} S^{-1} \\ &- S \begin{Vmatrix} 0 & 0 \\ 0 & I \end{Vmatrix} S^{-1} = S \begin{Vmatrix} 0 & 0 \\ 0 & Y \end{Vmatrix} S^{-1} \end{aligned}$$

follows

$$M^{(1)} = S \begin{Vmatrix} U & T \\ T & Y^{(1)} \end{Vmatrix} S^{-1} \qquad (U, V, T \text{ arbitrary})$$

and so

$$P_D P(I - M^{(1)}M) = S \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix} S^{-1} S \begin{Vmatrix} I & -VY \\ 0 & I - Y^{(1)}Y \end{Vmatrix} S^{-1} = S \begin{Vmatrix} 0 & 0 \\ 0 & I - Y^{(1)}Y \end{Vmatrix} S^{-1}.$$

Hence,

$$S \begin{Vmatrix} 0 \\ W^{n-k-1}(I - Y^{(1)}Y)Z(g^k x) \end{Vmatrix} = S \begin{Vmatrix} 0 & 0 \\ 0 & W^{n-k-1} \end{Vmatrix} \begin{Vmatrix} 0 & 0 \\ 0 & I - Y^{(1)}Y \end{Vmatrix} S^{-1} H(g^k x)$$

$$= S \begin{Vmatrix} 0 & 0 \\ 0 & R^{-1} \end{Vmatrix}^{n-k-1} S^{-1} S \begin{Vmatrix} \lambda N - I & 0 \\ 0 & \lambda R - I \end{Vmatrix}^{n-k-1} \begin{Vmatrix} 0 & 0 \\ 0 & I - Y^{(1)}Y \end{Vmatrix} S^{-1} H(g^k x)$$

$$= P_D^{n-k-1} Q^{n-k-1} P_D P(I - M^{(1)}M) H(g^k x),$$

where $H(x) = S \begin{Vmatrix} \Phi(x) \\ Z(x) \end{Vmatrix}$ is arbitrary. This implies that the right hand sides of (18) and (19) are equal.

Remark 3. Notice the analogy between (19) and the general solution of (7). Namely, if $P = I$, then $P_D = I$, $M = Q^n - I$, and (19) formally becomes

$$F(x) = \sum_{k=0}^{n-1} Q^{n-k-1}(I - M^{(1)}M)H(g^k x).$$

Remark 4. Combining the results of Theorems 1, 2 and 3 we say may that the general solution of the system (4) can be obtained provided that there exist $\lambda, \mu \in \mathbf{C}$ such that $\lambda A + \mu B$ is nonsingular.

Remark 5. The case when there is no $\lambda \in \mathbf{C}$ such that $\lambda A - B$ is nonsingular remains unsolved. It may happen, however, that the system (4) can then be solved directly, since A and B become rather special. As an illustration, we consider the case $m = 2$ in some detail. Let

$$A = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}, \quad B = \begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix}.$$

If $\lambda A - B$ is singulat, i. e. if $\det(\lambda A - B) = 0$ for all $\lambda \in \mathbf{C}$, we have $(ad - bc)\lambda^2 + (b\gamma + \beta c - a\delta - \alpha d)\lambda + (\alpha\beta - \beta\gamma) = 0$ for all $\lambda \in \mathbf{C}$. Since A and B are singular by supposition, we have $c = ka, d = kb, \gamma = l\alpha, \delta = l\beta$ for some $k, l \in \mathbf{C}$, and we are left with

$$(k - l)(a\beta - \alpha b) = 0.$$

If $k \neq l$, then for some $m \in \mathbf{C}$ we have $\alpha = ma$, $\beta = mb$, and the system (4) becomes

$$\varphi(gx) = m\varphi(x), \quad k\varphi(hx) = lm\varphi(x),$$

with $\varphi(x) = af_1(x) + bf_2(x)$, with the obvious general solution $\varphi(x) \equiv 0$. Hence, it is satisfied by all functions f_1, f_2 such that $af_1(x) + bf_2(x) = 0$.

If $k = l$, the system (4) reduces to the single equation

$$af_1(gx) + bf_2(gx) = \alpha f_1(x) + \beta f_2(x),$$

which is equivalent to $(a - \alpha)\varphi(x) + (b - \beta)\psi(x) = 0$, where $\varphi(x) = \sum_{k=0}^{n-1} f_1(g^k x)$, $\psi(x) = \sum_{k=0}^{n-1} f_2(g^k x)$ and is easily handled, since it remains to solve at most two linear cyclic equations.

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Tikveška 2
11000 Beograd

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