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SOME PROPERTIES OF THE QUASIASYMPTOTIC OF SCHWARTZ DISTRIBUTIONS PART II: QUASIASYMPTOTIC AT 0

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Abstract. We give the definition of the quasiasymptotic behaviour at 0 of Schwartz distributions from $\mathcal{D}'$ and compare this definition with the definition of the quasiasymptotic of tempered distributions at 0 [**2**].

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1. Definitions. The quasiasymptotic behaviour at 0 of tempered distributions was considered in [**8**] and [**2**]. First we reformulate the definition from [**2**].

Definition 1. Let $f \in \mathcal{S}'$ and $c(x),\ x \in (0,a),\ a > 0$, be a mearurable positive function. It is said that f has (in $\mathcal{S}'$) the quasiasymptotic at 0 with recpect to $c(1/k)$ if there is a $g \in \mathcal{S}', g \neq 0$, such that

$$\lim_{k\to\infty}\Big\langle \frac{f(x/k)}{c(1/k)},\ \varphi(x)\Big\rangle = \langle g(x),\ \varphi(x)\rangle, \quad \varphi \in \mathcal{S}. \tag{1}$$

In this case we write $f \sim^q g$ at 0 with respect to $c(1/k)$ (in $\mathcal{S}'$).

We extend this definition.

Definition 2. Let $f \in \mathcal{D}'$ and c be as in Definition 1. It is said that f has (in $\mathcal{D}'$) the quasiasymptotic at 0 with respect to $c(1/k)$ if there is a $g \in \mathcal{D}',\ g \neq 0$, such that

$$\lim_{k\to\infty}\Big\langle \frac{f(x/k)}{c(1/k)},\ \varphi(x)\Big\rangle = \langle g(x),\ \varphi \in \mathcal{D}. \tag{2}$$

In this case we write $f \sim^q g$ at 0 with recpect to $c(1/k)$ (in $\mathcal{D}'$).

Obviously, if $f \in \mathcal{S}'$ and $f \sim^q g$ at 0 with respect to $c(1/k)$ (in $\mathcal{S}'$) then $f \sim^q g$ at 0 with respect to $c(1/k)$ in $\mathcal{D}$.

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Theorem 1. *Let f and c satisfy conditions of Definition 2. Assume further that c is continuous. Then for some real number ν and some slowly varying function L at 0^+*

$$c(x) = x^\nu L(x),\ x \in (0,a).$$

Moreover, g is homogeneous with the order of homogeneity ν.

(Slowly varying functions are studied in [**4**]).

Proof. Let $f \in \mathcal{D}$ be such that $< g, \varphi > \neq 0$. For any $m > 0$ we have

$$\Big\langle \frac{f(mx/k)}{c(m/k)}, \varphi(x)\Big\rangle \to \langle g(x),\ \varphi(x)\rangle,\ k \to \infty,$$

$$\Big\langle \frac{f(mx/k)}{c(1/k)}, \varphi(x)\Big\rangle \to \langle g(mx),\ \varphi(x)\rangle,\ k \to \infty;$$

$$\left(\frac{c(m/k)}{c(1/k)}\right)\Big\langle \frac{f(mx/k)}{c(m/k)}, \varphi(x)\Big\rangle \to \langle g(mx),\ \varphi(x)\rangle,\ k \to \infty. \tag{3}$$

This implies that for any $m > 0$

$$\lim_{k\to\infty} \frac{c(m/k)}{c(1/k)} \text{ exists.} \tag{4}$$

From [**4**, 1.4] it follows that the limit function defined by (4) must be equal to $m^\nu, m > 0$, and that for that $\nu \in R$ and for some function L slowly varying at 0^+ one has

$$c(x) = x^\nu L(x),\ x \in (0,a). \tag{5}$$

Also, (3) implies that $g(mx) = m^\nu g(x),\ m > 0,\ x \in R$, which completes the proof.

This theorem directly implies:

Theorem 2. *Let f and c satisfy conditions of Definition 1 and let c be continuous. Then the assertion of Theorem 1 holds.*

Some obivous properties of the quasiasymptotic at 0 in $\mathcal{S}'$ are given in the next theorem.

Theorem 3. *Let $f \in \mathcal{S}'$ and $f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{S}'$). Then:* (i) *if $g^{(m)} \neq 0$ then $f^{(m)} \sim^q g^{(m)}$ at 0 with respect to $(1/k)^{\nu-m}L(1/k)$ (in $\mathcal{S}'$), $m \in N$,* (ii) *$x^m f(x) \sim^q x^m g(x)$ at 0 with respect to $(1/k)^{\nu+m}L(1/k)$ (in $\mathcal{S}'$) if $m \in N$ and $\nu \notin -N$;* (iii) *if $\nu \in -N,\ m \in N$ and $m <| \nu |$, then $x^m f(x) \sim^q x^m g(x)$ at 0 with respect to $(1/k)^{\nu+m}L(1/k)$ (in $\mathcal{S}'$).*

The same assertions hold for the quasiasymptotic at 0 in $\mathcal{D}'$

The quasiasymptotic at 0 is a local property of a distribution. Namely,

Theorem 4. *Let $f \in \mathcal{D}'$ and $f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{D}'$), and let $f_1 \in \mathcal{D}'$ be such that $f = f_1$ in some neighbourhood of zero. Then $f_1 \sim^q g$ at 0 with respect to $(1/k)^\nu \cdot L(1/k)$.*

Proof. Follows from the equality $\langle f(x),\ \varphi(kx)\rangle = \langle f_1(x),\ \varphi(kx)\rangle$ which holds for any $\varphi \in \mathcal{D}$ if $k > k_0(\varphi)$.

The some assertion holds for the quasiasymptotic at 0 (in $\mathcal{S}'$). This was proved in [**2**, Lemma 1.6].

THEOREM 5. *Let* $f \in \mathcal{S}'$, *resp.* $f \in \mathcal{D}'$, *and* $f \sim^q g$ *at* 0 *with respect to* $c(1/k)$ *(in* $\mathcal{S}'$, *resp. in* $\mathcal{D}'$*). Let* $\omega \in \mathcal{S}$, *resp.* $\omega \in \mathcal{E}$ *and*

$$\frac{\omega(x/k)}{c_1(1/k)} \to \omega_0(x) \ \text{ in } \mathcal{S}, \ \text{ resp. in } \mathcal{E}, \ \text{ as } k \to \infty,$$

where $c(x),\ x \in (0,a),\ a > 0$, *is a measurable positive function. Then* $f\omega \sim^q \omega_0 g$ *at* 0 *with respect to* $c(1/k)c_1(1/k)$ *(in* $\mathcal{S}'$, *resp. in* $\mathcal{D}'$*).*

Proof. Follows from [**7**, Y. I, p. 72., Théorème X].

2. Relations between two definitions. Let $f \in \mathcal{S}'$ and $f \sim^q g$ at 0 with respect to $c(1/k)$ (in $\mathcal{D}'$). The question is: Does the same hold in $\mathcal{S}'$? We shall prove in this section that for $c(1/k) = (1/k)^\nu L(1/k),\ k > 1/a$, the answer to the question is affirmative if $\nu > 0$ or if $0 \geq \nu > -1$ and L is bounded in some interval $(0,\eta),\ \eta > 0$. Otherwise the problem is still open.

Theorem 3(i) and [**2**, Lemma 1.7] directly imply the following:

THEOREM 6. *Let* $f \in \mathcal{S}'$ *and* $f = F^{(m)}$ *in some neighbourhood of* 0, *where* $m \in N_0$ *and* F *is a locally integrable function such that for some* $\nu > -1, L$ *and* $(C_+, C_-) \neq (0,0)$,

$$\lim_{x \to \pm 0} \frac{F(x)}{|\, x\,|^\nu\ L(|\, x\,|)} = C_\pm.$$

Then $f \sim^q g$ *at* 0 *with respect to* $(1/k)^\nu L(1/k)$ *(in* $\mathcal{S}'$*), where* $g = (C_+ x_+^\nu + C_- x_-^\nu)^{(m)}$, *and* $x_\pm^\nu = H(\pm x)\,|\, x\,|^\nu$ *(*H *is the Heaviside function).*

The following theorem is proved in [P].

THEOREM 7. *Let* $f \in \mathcal{S}, f \sim^q g$ *at* 0 *with respect to* $(1/k)^\nu L(1/k)$ *(in* $\mathcal{D}'$*), where* $\nu > 0$. *Then there is a continuous function* $F(x),\ x \in (-1,1)$ *and an* $m \in N_0$ *such that* $f = F^{(m)}$ *in* $(-1,1)$ *and*

$$\lim_{x \to \pm 0} \frac{F(x)}{|\, x\,|^{\nu+m}\ L(|\, x\,|)} = C_\pm. \ \text{for some } \ (C_+, C_-) \neq (0,0)$$

If $0 \geq \nu > -1$ and $L(x)$ is bounded in some interval $(0,\eta)$, the assertion holds as well.

Theorems 6 and 7 directly imlpy the following

THEOREM 8. *Let* f *satisfy conditions of Theorem 7. Then* $\sim^q g$ *at* 0 *with respect to* $(1/k)^\nu L(1/k)$ *(in* $\mathcal{S}'$*).*

For $\nu < 0$ we have a partial answer to the question above.

Let us denote by $\mathcal{Z}$ the space of Fourer transformations of elements from $\mathcal{D}$ supplied by the convergence structure transported from $\mathcal{D}(\mathcal{Z}=\mathcal{F}(\mathcal{D}))$. Let $f\in\mathcal{S}'$. We write $f\sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{Z}'$) if $g\in\mathcal{Z}'$, $g\neq 0$, and

$$\lim_{k\to\infty}\Big\langle\frac{f(x/k)}{(1/k)^\nu L(1/k)},\ \varphi(x)\Big\rangle=\langle g(x),\ \varphi(x)\rangle,\ \varphi\in\mathcal{Z}.$$

Using the Fourier Transformations and Theorem I (i) (Part I) one can easily obtain that $g\in\mathcal{S}'$.

THEOREM 9. *Let $f\in\mathcal{S}'$ and $f\sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{Z}'$) where $\nu<0$, and $\nu\notin -N$. Then $f\sim^q q$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{S}'$).*

Proof. Since for some $m\in N_0$ and some continuous function F of slow growth $f=F^{(m)}$, the Fourier transformation implies

$$(-i)^m x^m\hat{F}(x)\sim^q \hat{g}(x)\ \text{ at }\ \pm\infty\ \text{ with respect to }\ k^{-\nu-1}L_1(k)$$

in the sence of convergence in $\mathcal{D}'$ and thus, in the sense of convergence in $\mathcal{S}'$ (see Theorem I, Part I). $L_1(\cdot)=L(1/\cdot)$ is slowly varying at ∞. Let us put

$$\hat{F}_+(x)=\begin{cases}\hat{F}(x), & x>0\\ 0, & x\le 0\end{cases}\qquad \hat{F}_-(x)=\begin{cases}0, & x\ge 0\\ \hat{F}(x), & x<0\end{cases}$$

[**2**, Lemma 2.2] implies that for some $N\in\mathbf{N}$ and some $(C_+,C_-)\in\mathbf{C}^2,(C_+,C_-)\neq(0,0)$

$$x^{m+N}\hat{F}_\pm(x)\sim^q C_\pm f_{-\nu+N}(\pm x)\ \text{ at }\ \pm\infty\ \text{ with respect to }\ k^{-\nu-1+N}L_1(k).$$

Now [2, Lemma 2.3] implies, for $-\nu-m>0$

$$\hat{F}_\pm(x)\sim^q C_\pm f_{-\nu-m}(\pm x)\ \text{ at }\ \pm\infty\ \text{ with respect to }\ k^{-\nu-1-m}L_1(k). \tag{6}$$

and for $-\nu-m<0$.

$$\hat{F}_\pm(x)\sim^q C_\pm f_{-\nu+N}(\pm x)+\sum_{j=0}^p a_{j\pm}\delta^{(j)}(x)\ \text{ at }\ \pm\infty \text{ with respect to } k^{-\nu-1-m}L_1(k). \tag{7}$$

Using the Inverse Fourier transformation we obtain: for $-\nu-m>0$

$$F(t)\sim^q\mathcal{F}^{-1}(C_+f_{-\nu-m}(t)+C_-f_{-\nu-m}(-t))$$
$$\text{at }\ 0\ \text{ with respect to }\ (1/k)^{\nu+m}L(1/k)\ \text{ (in }\ \mathcal{S}');$$

for $-\nu-m<0$

$$F(t)\sim^q\mathcal{F}^{-1}(\tilde{C}_+f_{-\nu-m}(t)+\tilde{C}_-f_{-\nu-m}(-t)+\sum_{j=0}^p a_{j+}\delta^{(j)}(t)+$$
$$+\sum_{j-0}^p a_j-\delta^{(j)}(t))\ \text{ at 0 with respect to}$$
$$(1/k)^{\nu+m}L(1/k)\ \text{ (in }\ \mathcal{S}'),\ ((\tilde{C}+,\tilde{C}-)\neq(0,0).)$$

Now Theorem 3 (i) completes the proof.

The proof of Theorem 9 shows that if ν and m satisfy the condition $-\nu - m > 0$, then the assertion of Theorem 9 holds without the asypmtotion $\nu \notin -\mathbf{N}$.

At the end we give a theorem which is a consequence of Theorem 5 from part I.

THEOREM 10. *Let $f \in \mathcal{S}'$ such that $xf \sim^q g$ at 0 with respect to*

$$(1/k)^{\nu+1} L(1/k),\ \nu \in R \backslash (-\mathbf{N}) \ \textit{(in } \mathcal{S}').$$

Let $\varphi_0 \in \mathcal{D}$ such that $\hat{\varphi}_0(0) = 1$ and

$$\left\langle \frac{f(x/k)}{(1/k)^\nu L(1/k)},\ \hat{\varphi}_0(x) \right\rangle \to \langle g_0(x), \hat{\varphi}_0(x) \rangle \ \ \textit{as} \ \ k \to \infty$$

such that $g_0 \in \mathcal{S}'$ an $xg_0(x) = g(x)$, $x \in \mathbf{R}$. Then, $f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{S}'$) (g and g_0 are homogeneous of order $\nu + 1$ and ν, respectively).

REFERENCES

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