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A GEOMETRIC CHARACTERIZATION OF HELICODIAL SURFACES OF CONSTANT MEAN CURVATURE

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Abstract. We prove that a helicodial surface has constant mean curvature if and only if its principal axes make an angle constant with the orbits. Moreover, the arguments used lead to a simple proof of the fact that all helicodial surfaces with constant mean curvature H can be isometrically deformed, trough helicodial surfaces of the same H, into surfaces of revolution of the same H (Delaunay surfaces).

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1. Introduction. *A one parameter subgroup of the group of rigid motions of* E^3 is a diferentiable mapping $\gamma : \mathbf{R} \times E^3 \to E^3$ with the following properties:

a) The map $\gamma_t : E^3 \to E^3$ given by $x \to \gamma(t,x)$, $x \in E^3$, $t \in \mathbf{R}$ is a rigid motion,

b) $\gamma_t \circ \gamma_s = \gamma_{t+s}$; c) $\gamma_0 =$ the identity.

Let $x \in E^3$ have coordinates $x = (x^1, x^2, x^3)$. It may be shown that, possibly after a change of basis, any one parameter subgroup may be written either as

$$\gamma(t,x) = (x^1 \cos t + x^2 \sin t, -x^1 \sin t + x^2 \cos t, x^3 + bt),$$

where $-\infty < b < +\infty$ is constant, or as

$$\gamma(t,x) = (x^1, x^2, x^3) + t(0,0,1) = (x^1, x^2, x^3 + t).$$

In the former case, if $b \neq 0$, $\gamma(t,x)$ is called a *helicodial motion with axis the x^3-axis and pitch b*. The *orbit* $t \in \mathbf{R} \to \gamma(t,x) \in E^3$ of a point $x = (x^1, x^2, x^3)$ which does not lie on the x^3-axis is a helix. All such helices have the x^3-axis as common axis. If $b = 0$, $\gamma(t,x)$ is a *rotational motion about the* x^3-axis. The orbits of points not lying on the x^3-axis are circles having the x^3-axis as common axis. In the latter case, $\gamma(t,x)$ is called a *parallel translational motion in the direction of the x^3-axis*. All orits are straight lines parallel to the x^3-axis.

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A helicodial surface with axis the x^3-axis and pitch $b \neq 0$ is surface that is invariant by $\gamma(t,x)$ for all t. When $b=0$ the surface reduces to a *surface of revolution.* Finally, the translational motions generate the *cylinders.*

If we consider a curve $c(s)$ on any of these surfaces which intersects all the orbits without touching them, the surface can be parametrized by (s,t) as $\gamma(t,c(s))$.

The main result states:

THEOREM 4. *A helicodial surface has constant mean curvature if and only if its principal axes make an angle constant with the orbits.*

This is not true for the surfaces of revolution and the cylinders since, regardless of H, the orbits are principal curves.

In [**3**], an analytic parametrization of the helicoidal surfaces of constant mean curvature was exhibited. Also, in [**3**] was first shown that these surfaces can be isometrically deformed under preservation of the mean curvature and through helicoidal surfaces into Delaunay surfaces. Here, a simpler proof of this fact is given by making use of facts for general (not necessarily with H =constant) helicodial surfaces. The main tool for the proof of this part is the following:

THEOREM (O. Bonnet) (cf. [**1, 2, 4**]) *A surface of constant mean curvature in E^3, other than the plane and the sphere, can be isometrically deformed so that the mean curvature is preserved. During this deformation the principal directions rotate by a fixed angle, and for any fixed angle as rotation angle of the principal directions a surface of this isometric deformation is obtained.*

2. Some local surface theory. We consider a surface M^2 in E^3, orientable and sufficiently smooth. We consider a well defined field of orthonormal frames (x, e_1', e_2', e_3') over M^2, such that $x \in M$, and e_1, e_2 comprise an orthonormal basis of the tangent space of M at x. We have then

$$\eta_i = dx \cdot e_i', \quad \eta_{ij} = de_i' \cdot e_j', \quad \eta_{ij} = -\eta_{ji} (\text{so } \eta_{ii} = 0)$$

$d\eta_i = \sum_{j=1}^{3} \eta_{ij} \wedge \eta_j$ (1-st structural equation), $d\eta_{ij} \sum_{k=1}^{3} \eta_{ik} \wedge \eta_{kj}$ (2nd structural equation) where $1 \leq i,\ j \leq 3$. On $M^2, \eta_3 = 0$ so we have $\eta_{13} \wedge \eta_1 + \eta_{23} \wedge \eta_2 = 0$. So, by Cartan's Lemma we get

$$\eta_{13} = a\eta_1 + b\eta_2, \quad \eta_{23} = b\eta_1 + c\eta_2. \tag{1}$$

Then the mean and Gaussian curvatures of M^2 are

$$H = (a+c)/2, \quad K = ac - b^2.$$

We also have

$$d\eta_{12} = -K\eta_1 \wedge \eta_2 \qquad \text{(Gauss Equation) (GE).}$$

$$\left.\begin{aligned} d\eta_{13} &= \eta_{12} \wedge \eta_{23} = -bd\eta_2 + cd\eta_1 \\ d\eta_{23} &= \eta_{21} \wedge \eta_{13} = ad\eta_2 - bd\eta_1 \end{aligned}\right\} \quad \text{(Codazzi-Mainardi Equations) (CME).}$$

A given Riemannian surface can be realized in E^3 if the CME are satisfied.

We now let

$$e_1 = \cos\psi e_1 - \sin\psi e_2, \;\; e_2 = \sin\psi e_1 + \cos\psi e_2$$

be the principal frame of M^2. For this frame the function b defined by (1) is zero and a, c are the principal curvatures.

In the sequel, we consider M^2 with no umbilic points. We may then assume for the principal curvatures k_1, k_2 of M^2 that $k_1 > k_2$ and we put $J = k_1 - k_2 > 0$. We will show that the CME are equivalent to:

$$dH = H_1\eta_1 + H_2\eta_2, \quad \text{(thus defining} \quad H_1, H_2) \tag{2}$$

$$\begin{aligned} d\psi = &- \cos\, 2\psi[H_2J^{-1}\eta_1 + H_1J^{-1}\eta_2] = \sin\, 2\psi[-H_1J^{-1}\eta_1 + H_2J^{-1}\eta_2] + \\ &+ 1/2 * d\log\, J + \eta_{12}, \end{aligned} \tag{3}$$

where * is the Hodge operator whose action on the 1-forms is described by

$$*\eta_1 = \eta_2, \quad *\eta_2 = -\eta_1 \qquad (*^2 = -1).$$

Proof. The principal coframe is

$$\omega_1 = \cos\psi\eta_1 - \sin\psi\eta_2, \;\; \omega_2 = \sin\psi\eta_1 + \cos\psi\eta_2.$$

Exterior differentiation gives

$$d\omega_1 = (-d\psi + \eta_{12}) \wedge \omega_2,\, d\omega_2 = \omega_1 \wedge (-d\psi \wedge \eta_{12}).$$

Thus, the connection form assocated to the principal coframe is $\omega_{12} = -d\psi + \eta_{12}$, so that $d\psi = -\omega_{12} + \eta_{12}$. We then need to show that

$$\omega_{12} = \cos\, 2\psi[H_2J^{-1}\eta_1 + H_1J^{-1}\eta_2] - \sin\, 2\psi[-H_1J^{-1}\eta_1 + H_2J^{-1}\eta_2] - 1/2 * d\log\, J.$$

We work with ω_1, ω_2 and the associated forms $\omega_{12},\ \omega_{13},\ \omega_{23}$. We have that $\omega_{13} = k_1\omega_1$ and $\omega_{23} = k_2\omega_2$. We put $\omega_{12} = p\omega_1 + q\omega_2$. Then the equations $d\omega_{13} = \omega_{12} \wedge \omega_{23},\ d\omega_{23} = \omega_{21} \wedge \omega_{13}$ imply:

$$[dk_1 - (k_1 - k_2)p\omega_2] \wedge \omega_1 = 0,\, [dk_2 - (k_1 - k_2)q\omega_1] \wedge \omega_2 = 0.$$

We have $J = k_1 - k_2 > 0$ and we set $dH = d(k_1 + k_2)/2 = u\omega_1 + v\omega_2$. Then we get

$$dk_1 = (2u - Jq)\omega_1 + Jp\omega_2, \;\; dk_2 = Jq\omega_1 + (2v - Jp)\omega_2.$$

By subtracting the second relation from the first and dividing by J we have

$$d\log\, J = 2uJ^{-1}\omega_1 - 2vJ^{-1}\omega_2 + 2(-q\omega_1 + p\omega_2).$$

But $-q\omega_1 + p\omega_2 = *\omega_{12}$, thus we obtain

$$\omega_{12} = vJ^{-1}\omega_1 + uJ^{-1}\omega_2 - 1/2 * d\log J.$$

Now by comparing dH given by (2) and dH given by $dH = u\omega_1 + v\omega_2$, with ω_1, ω_2 expressed in terms of η_1, η_2 we get

$$u = H_1 \cos\,\psi - H_2 \sin\,\psi,\;\; v = H_1 \sin\,\psi + H_2 \cos\,\psi.$$

Now convert the last expression for ω_{12} to the desired final form by inserting the above relations for u, v and by again rewriting ω_1, ω_2 in terms of η_1, η_2.

The converse direction of the equivalence follows by reversing the process of this computation □

Remark 1. The Theorem of 0. Bonnet quoted in section 1 follows from (3) and the fundamental theorem of surfaces. With H constant on M^2 we get from (3) that

$$d\psi = -1/2 * d\log\,J + \eta_{12}.$$

Thus if some ψ satisfies this equation, so does ψ+constant, and the fundamental theorem of surfaces applies for each new ψ.

3. Some facts about Helicodial Surfaces. The geometry of a helicodial surface allows us to parametrize it by (s, t), where

s =arc-length of curves orthogonal to orbits measured from a fixed orbit,

t =time along orbits from a fixed $t = t_0$, (see also [**3**]). Then the curves t =constant are carried along the orbits by the motion. They remain orthogonal to the orbits and foliate the surface. So an orthonormal frame (e_1', e_2') is determined along these coordinate curves with e_2' tangent to the orbits. The corresponding coframe may be written as

$$\eta_1 = ds, \quad \eta_2 = q(s)dt \qquad (q \text{ depends only on } s).$$

Thus,

$$\eta_{12} = \frac{q'(s)}{q(s)}\eta_2 = \mu(s)\eta_2.$$

Hence the η_1-curves are geodesics and η_2-curves (orbits) have geodesic curvature equal to

$$\mu(s) = \frac{d}{ds}\log(\mid q(s) \mid).$$

Along each orbit a, c, μ, ψ are constant. So, in this case we get $H_2 = 0$. Also if we put $dJ = J_1\eta_1 + J_2\eta_2$, we get $J_2 = 0$ and $d\log\,J = J_1 J^{-1}\eta_1$. Hence relation (3) becomes

$$d\psi = -\cos\,2\psi[H_1 J^{-1}\eta_2] - \sin\,2\psi[H_1 J^{-1}\eta_1] + 1/2 J_1 J^{-1}\eta_2 + \mu\eta_2.$$

Since $\psi = \psi(s)$, this implies

$$\frac{d\psi}{ds} = -\sin 2\,\psi\frac{dH/ds}{J},\; \mu = \cos\,2\psi\frac{dH/ds}{J} - \frac{1}{2}\frac{dJ/ds}{J}. \tag{4}$$

By direct computation or by well known facts about curves on surfaces, we get the following:

The space curvature of orbits is

$$\sqrt{\mu^2 + (k_1 \cos^2 \psi + k_2 \sin^2 \psi)^2} \;\text{ or }\; \sqrt{\mu^2 + (k_1 \sin^2 \psi + k_2 \cos^2 \psi)^2},$$

and the space torsion of orbits is $\pm(k_1 - k_2) \sin \psi \cos \psi$, (5) for the respective cases when e_1 is the major and minor principal direction.

Finally, we show that a helicodial surface of constant mean curvature is free of umbilic points. This fact allows us to apply the previous local theory everywhere to such a surface.

Proof. Let M^2 be an oriented connected surface in E^3 with unit normal vector field v. We consider isothermal parameters (x_1, x_2) on M^2. If M^2 has constant mean curvature then the (locally defined) function

$$f(z) = (b_{11} - b_{22} - 2ib_{12}(z), \quad z = x_1 + ix_2$$

is known to be holomorphic in z; here $b_{ij} = -\langle \nabla_{\partial/\partial x_i} v, \partial/\partial x_j \rangle$, $1 \leq i,\ j \leq 2$, are the components of the second fundamental form of M^2. The zeros of $f(z)$ are exactly the umbilic points of M^2. So, if an umbilic point is not isolated then $f(z) \equiv 0$ and thus M^2 is totally umbilical; i.e., a piece of a sphere or a plane.

Now if we assume that a point on a helicodial surface is unbilic then all points belonging to the same orbit (helix) are umbilics and therefore that point is not isolated. So, when H is constant umbilic points connot exist: $\square$

4. Proof of the results. Looking at equation (4), we see that if $\psi \not\equiv$ multiple of $\pi/2$ then:

$$\psi \equiv \text{constant if and only if } H \equiv \text{constant}.$$

If ψ $\equiv$multiple of $\pi/2$ then, by (5) we have that orbits are plane curves. This happens only if the surface is a surface of revolution or cylinder. This finiches the proof of the geometric characterization claimed. $\square$.

Next, let us consider a helicodial surface M^2 invariant under the motion γ with constant H and its deformation guaranteed by the Theorem of 0. Bonnet (section 1). Let N^2 be a surface in this deformation, so that there is an isometry $f : M^2 \to N^2$ which is onto, preserves H (and hence preserves the principal curvatures), and rotates the principal frame by a fixed angle. It is easy to see that

$$f \circ \gamma \circ f^{-1}(y, t) :, = f[\gamma(f^{-1}(y), t)], \quad y \in N^2, \quad t \in \mathbf{R},$$

forms a one-parameter group of isometries (one checks that (b) and (c) hold), under which N^2 is invariant. Also these isometries preserve the principal curvatures and directions of N^2, hence they extend to rigid motions of E^3 (and thus property (a) is valid as well). So, N^2 is invariant under a one-parameter subgroup of rigid motions of E^3. The surface of revolution (Delaunay surface in this case) is obtained when, by rotating the principal frame, ψ becomes a multiple of $\pi/2$. $\square$

Remark 2. The Delaunay surfaces of a given constant mean curvature form a one-parameter family of surfaces. Thus the helicodal surfaces of a given constant mean curvature form a two-parameter family. The extra parameter is ψ.

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