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ALL REPRODUCTIVE SOLUTIONS OF FINITE EQUATIONS

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Summary. An equation, the solution set of which is a subset of a given finite set, is called a *finite equation.* Applying some kind of algebraic structure we effectively determine all reproductive solutions of such equations (*Theorem* 1 and *Theorem* 2).

1. Let E be a given non-empty set and f: $E \to E$ a given function. An x-equation

$$f(x) = x \tag{1}$$

is called *reproductive* [4] if the function f satisfies the identity

$$f(f(x)) = f(x) \tag{2}$$

All solutions of a reproductive equation can be found in a trivial way. Namely, the formula

$$x = f(p) \quad (p \text{ is an arbitrary element of } E) \tag{3}$$

determines all solution of (1) provided this equation is a reproductive one. The formula is an example of the so-called general reproductive solutions formula ([2], [3], [7]).

Next, any x-equation

$$eq(x) \quad \begin{array}{l}(x \text{ is an unknown element of } E;\ eq \text{ is a given}\\ \text{unary relation of } E)\end{array} \tag{4}$$

which has at least one solution is equivalent to a reproductive equation [4].

Accordingly, to solve a given x-equation (4) it suffices to find any reproductive equation equivalent to (1). In this paper we are concerned with finding all such reproductive equations in case of a finite equation.

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2. Let $B=\{\sigma_0,\sigma_1,\dots,\sigma_n\}$ be a given set of $n+1$ elements and $S=\{0,1\}$. Define the operation x^y by

$$x^y=\begin{cases}1, & \text{if } x=y\\ 0, & \text{otherwise}\end{cases}\qquad (x,y\in B\cup S) \tag{5}$$

The standard Boolean operations $+$ and $\cdot$ ("or" and "and") are described by the following tables

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

Extend these operations to partial operations of the set $B\cup S$ in the following way

$$x+0=x,\ \ 0+x=x,\ \ x\cdot 0=0,\ \ 0\cdot x=0,\ \ x\cdot 1=x,\ \ 1\cdot x=x \quad (x\in B\cup S) \tag{6}$$

We consider the following x-equation

$$s_0\cdot x^{b_0}+s_1\cdot x^{b_1}+\cdots+s_n\cdot x^{b_n}=0 \tag{7}$$

where $s_i\in\{0,1\}$ are given elements and $x\in B$ is unknown. Obviously the equation (7) is possible iff the condition

$$s_0\cdot s_1\cdots s_n=0 \tag{8}$$

holds.

In the sequel, assuming the condition (8), we are going to determine all reproductive solutions of the equation (7).

First we introduce the following definition.

Let $(\sigma_0,\sigma_1,\dots,\sigma_n)\in S^{n+1}$ be any element. Then the set $Z(\sigma_0,\dots,\sigma_n)$ ("the zero-set of $(\sigma_0,\dots,\sigma_n)$") is defined as folows

$$b_i\in Z(\sigma_0,\dots,\sigma_n)\Leftrightarrow \sigma_i=0 \qquad (i=0,1,\dots,n) \tag{9}$$

For instance, if $n=3$ we have

$$Z(1,0,1,0)=\{b_1b_3\},\ Z(1,1,1,1)=\emptyset,\ Z(0,0,0,0)=\{b_0,b_1,b_2,b_3\}$$

Let now $s_0,\dots,s_n$ be any elements of S satisfying the condition (8). With respect to the sequence $s_0,\dots,s_n$ we define the so-called *repro-function*[1] $A:\ B\to B$. This is any function defined by a certain formula of the form

$$A(x)=A_0(s_0,\dots,s_n)x^{b_0}\cdots+A_n(s_0,\dots,s_n)x^{b_n} \tag{10}$$

where each coefficient $A_k(s_0,\dots,s_n)$ is determined by some equality of the form

$$A_k(s_0,\dots,s_n)=b_ks_k^0+\sum_{\sigma_k\neq 0,\ \sigma_0\cdots\sigma_n=0}F_k(\sigma_0,\dots,\sigma_n)s_0^{\sigma_0}\cdots s_n^{\sigma_n} \tag{11}$$

[1]As a matter of fact, A is a function of the type $A:\ S^{n+1}\times B\to B$.

assuming that coefficients $F_k(\sigma_0, \dots, \sigma_n) \in B$ satisfy the condition

$$F_k(\sigma_0, \dots, \sigma_n) \in Z(\sigma_0, \dots, \sigma_n) \tag{12}$$

For instance, if $n = 2$, $k = 1$ the equality (11) reads

$$A_1(s_0, s_1, s_2) = b_1 s_1^0 + F_1(0,1,0)s_0^0 s_1^1 s_2^0 + F_1(0,1,1)s_0^0 s_1^1 s_2^1 + F_1(1,1,0)s_0^1 s_1^1 s_2^0,$$

where the coefficients $F_1(0,1,0)$, $F_1(0,1,1)$, $F_1(1,1,0)$ can be any elements of B satisfying the conditions (of type (12))

$$F_1(0,1,0) \in \{b_0, b_2\}, \;\; F_1(0,1,1) = b_0, \;\; F_1(1,1,0) = b_2$$

Note that generally, according to the condition $s_0 \cdots s_n = 0$, there exists at least one repro-function with respect to the sequence $s_0, \dots, s_n$.

THEOREM 1. *Let $s_0 \cdots s_n = 0$, then the equation of the form*

$$x = A(x)$$

is a reproductive equation and equivalent to the equation (7) *if and only if A is a repro-function.*

Proof. Denote the equation (7) by $g(x) = 0$. Firstly, we prove the following fact

(p) Let $A(x)$ be determined by a certain equality of type (10) assuming only that $A_0(s_0, \dots, s_n)$, $\dots$, $A_n(s, \dots, s_n) \in B$. Then the implication $g(x) = 0 \Rightarrow x = A(x)$ holds if and only if for each coefficient $A_k(s_0, \dots, s_n)$ an equality of the form (11) is satisfied, where $F_k(\sigma_0, \dots, \sigma_n)$ can be any elements[2] of B.

The proof immediately follows from the following equivalences

$$\begin{aligned}(p) &\Leftrightarrow (\forall x \in B)[s_0 x^{b_0} + \cdots + s_n x^{b_n} = 0 \Rightarrow x = A_0(s_0, \dots, s_n)x^{b_0} + \cdots + \\ &\qquad + A_n(s_0, \dots, s_n)x^{b_n}] \\ &\Leftrightarrow (\forall k \in \{0, \dots, n\})(s_k = 0 \Rightarrow A_k(s_0, \dots, s_n) = b_k) \\ &\Leftrightarrow A_k(s_0, \dots, s_n) \text{ is determined by means of a certain equality (11), where} \\ &\qquad F_k(\sigma_0, \dots, \sigma_n) \text{ may be any elements of } B.\end{aligned}$$

Next we introduce the condition

$$(\forall x \in B) s_0 (A(x))^{b_0} + \cdots + s_n (A(x))^{b_n} = 0. \tag{q}$$

Obviously the sentence "$x = A(x)$ *is a reproductive equation, equivalent to the equation* $f(x) = 0$" is logiccally equivalent to the conjuction $(p) \wedge (q)$. Accordingly, the remaining part of the proof reads:

$x = A(x)$ is a reproductive equation, equivalent to the equation $f(x) = 0$

[2] Thus condition (8) is not assumed.

$\Leftrightarrow$ $(p) \wedge (q)$

$\Leftrightarrow$ $(\forall x \in B) s_0(A(x))^{b_0} + \cdots + s_n(A(x))^{b_n} = 0$, and $A(x)$ is determined by means of some equalities of the form (10), (11), where $F_k(\delta_0, \ldots, \delta_n)$ are certain elements of B

$\Leftrightarrow$ $(\forall x \in B)\ s_0(A_0(s_0, \ldots, s_n)x^{b_0} + \cdots + A_n(s_0, \ldots, s_n)x^{b_n})^{b_0} + \cdots +$
$$+ s_n(A_0(s_0, \ldots, s_n)x^{b_0} + \cdots + A_n(s_0, \ldots, s_n)x^{b_n})^{b_n} = 0$$
and $A_k(s_0, \ldots, s_n)$, with $k \in \{0, \ldots, n\}$, are determined by some equalities of the form (11).

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\})\ s_i A_k^{b_i}(s_0, \ldots, s_n) = 0$ and $A_k(s_0, \ldots, s_n)$, with $k \in \{0, \ldots, n\}$, are determined by some equalities of the form (11). This part of the proof is based on the following general facts: If $a_0, \ldots, a_n$, b are any elements of B then:

1° $(a_0 x^{b_0} + \cdots + a_n x^{b_n})^b = a_0^b x^{b_0} + \cdots + a_n^b x^{b_n}$ (for all $x \in B$)

2° $(\forall x \in B)\ a_0 x^{b_0} + \cdots + a_n x^{b_n} = 0 \Leftrightarrow (\forall i \in \{0, \ldots, n\}) a_i = 0$

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\})\ s_i \cdot (\sum_{\sigma_k \neq 0,\ \sigma_0 \cdots \sigma_n = 0} F_k^{b_i}(\sigma_0, \ldots, \sigma_n) s_0^{\sigma_0} \cdots s_n^{\sigma_n}) = 0$, where $F_k(s_0, \ldots, s_n)$ are certain elements of B. We have used the equality of the form (11) and the identity $s_i \sigma_k^{b_i} s_k^0 = 0$

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\})$
$$\Big(\sum_{(\sigma_0, \ldots, \sigma_n) \in S^{n+1}} \sigma_i s_0^{\sigma_0} \cdots s_n^{\sigma_n}\Big) \cdot \sum_{\sigma_k \neq 0,\ \sigma_0 \cdots \sigma_n = 0} F_k^{b_i}(\sigma_0, \ldots, \sigma_n) s_0^{\sigma_0} \cdots s_n^{\sigma_n} = 0$$
(For the identity $s_i = \sum_{(\sigma_0, \ldots, \sigma_n) \in S^{n+1}} \sigma_i s_0^{\sigma_0} \cdots s_n^{\sigma_n}$ holds).

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\}) \sum_{\sigma_k \neq 0,\ \sigma_0 \cdots \sigma_n = 0} \sigma_i F_k^{b_i}(\sigma_0, \ldots, \sigma_n) s_0^{\sigma_0} \cdots s_n^{\sigma_n} = 0$

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\})\ (\forall \sigma_0, \ldots, \sigma_n \in S)(P \Rightarrow \sigma_i F_k^{b_i}(\sigma_0, \ldots, \sigma_n) = 0)$
where the condition $\sigma_k \neq 0$, $\sigma_0 \cdots \sigma_n = 0$ is denoted by P.
From $F_k \in B$ it follows that $(\forall k)(\exists j) F_k = b_j$. Hence we conclude the equality $F_k = b_{\varphi(k)}$ where $\varphi : \{0, \ldots, n\} \to \{0, \ldots, n\}$ is a certain function[3].

$\Leftrightarrow$ $(\forall \sigma_0, \ldots, \sigma_n \in S)(\forall i,\ k \in \{0, \ldots, n\}(P \Rightarrow \sigma_i b_{\varphi(k)}^{b_i} = 0)$

$\Leftrightarrow$ $(\forall \sigma_0, \ldots, \sigma_n \in S)(\forall k \in \{0, \ldots, n\}(P \Rightarrow \sigma_{\varphi(k)} = 0)$
For $x \neq y \Rightarrow x^y = 0$.

$\Leftrightarrow$ $(\forall \sigma_0, \ldots, \sigma_n \in S)(\forall k \in \{0, \ldots, n\})(P \Rightarrow b_{\varphi(k)} \in Z(\sigma_0, \ldots, \sigma_n))$
Using definition (9).

$\Leftrightarrow$ $(\forall \sigma_0, \ldots, \sigma_n \in S)(\forall k \in \{0, \ldots, n\})(P \Rightarrow F_k \in Z(\sigma_0, \ldots, \sigma_n))$

$\Leftrightarrow$ A is a repro-function

[3] φ also depends on $\sigma_0, \ldots, \sigma_n$.

The proof is complete.

From Theorem 1 immediately follows the following result on the reproductive solutions.

THEOREM 2. *If* (7) *is a possible equation then a formula*

$$x = A(p) \qquad (p \text{ is any element of } B)$$

represents a general reproductive solution of the equation (7) *if and only if the function A is a repro-function.*

Example 1. Let $B = \{0, 1, 2\}$. Consider the x-equation

$$s_0 x^0 + s_1 x^1 + s_2 x^2 = 0 \qquad (s_i \text{ are given and } x \text{ is unknown})$$

This equation is possible if and only if $s_0 s_1 s_2 = 0$. Any general reproductive solution has the following form

$$x = A_0 p^0 + A_1 p^1 + A_2 p^2 \tag{14}$$

where A_i are defined by

$$A_0 = (1 \text{ or } 2) s_0^1 s_1^0 s_2^0 + 1 s_0^1 s_1^0 s_2^1 + 2 s_0^1 s_1^1 s_2^0, \quad A_1 = 1 s_1^0 + (0 \text{ or } 2) s_0^0 s_1^1 s_2^0 + 2 s_0^1 s_1^1 s_2^0$$

$$A_2 = 2 s_2^0 + (0 \text{ or } 1) s_0^0 s_1^0 s_2^1 + 1 s_0^1 s_1^0 s_2^1$$

In these equalities a symbol of the form (p or q) denotes an element which may be p or q. Consequently there are exactly 8 formulas of the form (14).

3. Now we state an aplication of Theorems 1 and 2. Let n be a given natural number and $B = S^n$. Then according to the definition (5) we have the following identity

$$(x_1, \dots, x_n)^{(i_1, \dots, i_n)} = x_1^{i_1} \cdots x_n^{i_n}$$

In connection with it the equation of the type (7) may be written in the following form

$$\sum a_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n} = 0 \tag{15}$$

where $a_{i_1 \dots i_n} \in S$ are given, and $x_j \in S$ are unknown elements. In other words, (14) is the standard Boolean equation in $x_1, \dots, x_n$. Theorems 1 and 2 can be directly applied to the equation (15); consequently one effectively finds all general reproductive solutions of it.

REFERENCES

[1] M. Božić, *A note on reproductive solutions*, Publ. Inst. Math. (Beograd) **19** (**33**) (1975), 33-35.

[2] L. Löwenheim, *Über das Auflösungs problem im logischen Klassen-kalkul.* Sitzungsber, Berl. Math. Geselschaft **7** (1980), 89–94.

[3] L. Löwenheim, *Über die Auflössung von Gleichungen im logischen Gebietenkalkul.* Math. Ann. **68** (1910), 169–207.

[4] S. B. Prešić, *Une classe d'équations matricielles et l'équation functionnele* $f^2 = f$, Publ. Inst. Math. (Beograd) **8** (**22**) (1968), 143–148.

[5] S. B. Prešić, *Une méthode de resolution des équations dont toutes les solutions appartiennent à un ensemble fini*, C.R. Acad. Sci. Paris Ser. A **272** (1971), 654–657.

[6] S. B. Prešić, *Ein Satz über reproductive Lösungen*, Publ. Inst. Math. Belgrade **14** (**28**) (1973), 133–136.

[7] S. Rudeanu, *Boolean Functions and Equations*, North-Holland, Amsterdam & Elsevier, New York, 1974.

[8] S. Rudeanu, *On reproductive solutions of arbitrary equations*, Publ. Inst. Math. (Beograd) **24** (**38**) (1978), 143–147.

[9] D. Banković, *On general and reproductive solutions of arbitrary equations*, Publ. Inst. Math. (Beograd) **26** (**40**), (1979), 31–33.

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