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RAMIFICATION HYPOTHESIS AGAIN

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Summary. To the RH (Ramification Hypothesis = Proposition 1 in Kurepa 1935:2,3 p. 130) we join here proposition $P_0'(s, 3:2)$, $P_{18}, P_{19}, \ldots, P_{45}$, each equivalent to RH; we stress in particular $P_{18} := P_s$: For every branching tree T the width p_sT^2 of the cardinal square of T equals p_sT. (s. 1:0) and is attained (s. № 3).

0. Introduction.

0:0. In my doctoral dissertation 1935:2, 3 p. 130 the following ramification hypothesis (RH) was formulated (cf. also 1936:1).

P_1 For any tree T the number bT is attained in the sense that T contains a degenerate subset of cardinality bT ($bT := \sup pD$, D running trought the system P_DT of all degenerate subsets of T; $pD :=$ power of D; an ordered set S is quoted as degenerate if for every $x \in S$ the corresponding cone $S(x)$ consisting of all elements of S, each comparable to x, is a subchain of $(S, \leq)$).

0:1. My dissertation 1935:2, 3 contains following 15 pairwise equivalent propositions:

$$P_0,\ P_1,\ P_2,\ P_2',\ P_3, \ldots,\ P_{12},\ P_b,$$

(s, 1935:2, 3 pp. 130–132 for $P_1,\ P_2, \ldots,\ P_{12}$; p. 130_{5-1} for P_2' and № 9:$4^{5,6}$ p. 9:3 for P_0 and № 11:5 p. 111 for P_b).

0:1:0. P_0: *Every infinite completely ramified sequence S contains an antichain of power $p\gamma S$ where γT denoted the height of T* (a tree T was called a sequence if every $x \in T$ is such that $\gamma T(x) = \gamma T$; a T was quoted as completely ramified provided for every $x \in T$ one has $T(\cdot, x) = T(\cdot, y)$ for at least one $y \in T \setminus \{x\}$, where $T(\cdot, x) := \{z : z < x,\ z \in T\}$.

0:1:1. P_b: *For every tree T unless the height γT is inaccessible, the number bT is attained in T* (s. № 5 p. 111 in Kurepa 1935:2, 3).

0:2. We stress as very handlable the following

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P_2 Reduction Principle (RP): *Every infinite tree T is equinumerous to a degenerate subtree.*

One speaks for short: T is D-reflexive, in the sense of the following.

0:3. *Definition.* A graph $(V,\, R)$ is quoted as D-reflexive provided V is equinumerous to a direct sum of a system of complete subgraphs. The word "D-reflexive" replaces the word "normal" used in my Thesis (cf: Thesis, № 11.1, p. 105).

0:4. Afterwards, I formulated other propositions: P_{13} (s. 1977:1 № 5:1 with references 1950:8, 1952:8, 1953:11, 1953:12), P_{14} (s. 1977:1 № 7:7), P_{15} (1977:1 № 7:8), P_{16} (1977:1 № 3:1), P_{17} (v. 1977:1 № 3:2), $\bar{P}_{17}$ (dual of P_{17}; s. 1977:1 №3:3) each equivalent to RH.

Consequently, one has 21 pairwise equivalent propositions $P_0, P_1, \ldots, P_{17}, P_1, \ldots, P_{17}, P_2', \bar{P}_{17}, P_b$.

0:5. *Inaccessible variations.* If P denotes any of these 21 propositions, let $P(i)$ denote the corresponding proposition restricted to the case that the corressponding power be inaccesible (= initial limit regular alef). So one gets 21 propositions $P_0(i),\ P_1(i), \ldots,\ \bar{P}_{17}(i),\ P_b(i)$.

0:6. So e.g. we have

$P_5(i)$ In every linearly ordered set L of an inaccessible cellularity there is a disjoint family of cardinality sep $L := dL$ of open non empty intervals of L (cf. 1977:1 № 2:6 where instead of "inaccessible cardinality" should be read "inaccessible cellularity").

0:7. For a topological space S the density number is $dS := \inf\{pX;\, X \subset S,\, X$ is everywhere dense in $S\}$. The cellularity of S is $cS := \sup\{pD : D$ consists of pairwise disjoint open sets $\subset S\}$.

1. Some consequences of RH.

I had the opportunity to formulate some interesting consequences of the RH like: ReH, P_{15}', MATH, P_5^0; $L(i)$:

1:0. Rectangle hypothesis. (ReH) *Every tree T satisfies $pT \le p_cT \cdot p_sT$* (s. 1964:7; 1977:1 № 7:2) *where p_cT (resp. P_sT) is* $\sup pX$, *X running through the system of all subchains (subantichains) of T.*

1:1. Proposition P_{15}': *Every tree is the union of p_sT* (s. 1:0) *subchains of T* (s. 1963:3 Theorem 3:3) *i.e. $p_sT = sT$, where $s(E,\le)$ denotes the star number of $(E,\le)$ ($s(E,\le)$ is the minimal number of subchains of $(E,\le)$ exhausting E*(s. 1963:3 № 1:1);

1:2. MATH (Maximum Antichain Tree Hypothesis): *Every tree contains a maximum antichain i.e. in every T the number $p_sT$is attained* (s. 1977:1 № 8:1, 1987:1 cf. also 1987:1).

1:3. Proposition P_5^0. *Every infinite chain L satisfies $cL = dL$* (s. 1977: № 2:3);

1:4. PROPOSITION $L(i)$. *Every ordered chain of inaccessible separability contains a maximum disjoint system of open sets* (v. 1977:1 № 8:6 where instead of $L_1(i)$ should be $L(i)$).

1:5. THEOREM MATH $\Leftrightarrow L(i)$ (s. 1977:1 Theorem 8:5). One has

1:6. THEOREM $ReH \Leftrightarrow P'_{15}$ (s. 1977:1 Theorem 3:3)

1:7. THEOREM $P_5 \Leftrightarrow P_5^0 \,\&\, P_5(i)$ (announced in 1977:1 as Th 2:5).

Proof. The $\Rightarrow$ – part of the statement being obvious, let us prove the $\Leftarrow$ – part. Let L be any infinite ordered chain; if $p_2L := \operatorname{cel} L$ is limit and regular, then, by assumption $P_5(i)$, P_5 holds; if cel L is regular and isolated, then cL is attained and, by P_5^0, equals sep L; thus L has a disjoint system of power sep L of intervals. The case when cel L is singular was considered explicitly in 1987:1 as Theorem 0:11 and is implied by the theorem 3 p. 110 in Kurepa 1935:2,3.

2. Some equivalences

2:0. THEOREM $P_0^5 \Leftrightarrow ReH$ (cf. 1:0, 1:3).

Proof. Part $\Rightarrow$. In opposite case there would exist a tree T such that $pT > p_cT \cdot p_sT$. One could assume without restriction that the rank or height γT is a regular initial ordinal and that T is a sequence (i.e. $\gamma T(x) = \gamma T$ for every $x \in T$; where $T(x) := \{y : y \in T$ and y is comparable to $x\}$ and that if $x \in R_{\alpha+1}T$ then there are infinitely many members y of T such that $T(\cdot, x) = T(\cdot, y)$. If then one orders totally every node N of T in such a way that if γN (defined by $N \subset R_{\gamma T}T$) is isolated the chain $(N, \leq_N)$ has no first element, then the natural ordering of T which extends $(T, \leq)$ as well as $(T, \leq_N)$ for every node N yields an ordered chain L (s. 1935 · 2,3 p.127); one verifies that $p_sT = cL$, $dL = pT$; consequently, one would have $cL = p_sT < pT = dL$ thus $cL < dL$, contrarily to the assumption P_5^0.

Part $\Leftarrow$. In the opposite case, there would exist an infinite chain $(L, \leq)$ such that cel $L < dL$. Let D be a complete dyadic atomization i.e. a complete bipartition of $(L, \leq)$ (s. 1935:2,3 p. 114); then the system F of members of D wich are segments of $(L, \leq)$ of power > 1 is such that $pF = dL$; the height γF of $(F, \supset)$ should be the initial ordinal $\beta := \omega_{(dL)}$.[1] Now, γF is not attained (in the opposite case, there would exist a $\beta-$sequence of strictly increasing intervals $I_n \in F (n < \beta)$; there is no restriction to suppose that L has no gap and therefore $\inf I_n$, $\sup I_n \in L$; consequently one of the sequences $\inf I_n (n < \beta)$, $\sup I_n (n < \beta)$, should be of power dL. Therefore at least one of the systems

$$(\inf I_n,\ \inf I_{n+1}) \cap L,\ (\sup I_n,\ \sup I_{n+1}) \cap L \quad (n < \beta)$$

would yield a disjoint system of cardinality dL of non void intervals of L, contrarily to the assumption $cL < dL$.

2:1. THEOREM. *The propositions P_5^0, P'_{15}, ReH are pairwise equivalent* (cf. 1:3, 1:1, 1:0).

[1]For a cardinal n one denotes by $\omega_{(n)}$ the first ordinal of power n.

This is implied by 1:6, 2:0.

2:2. LEMMA. $P_5(i) \Rightarrow L(i)$.

Proof. Let L be an ordered chain such that sL be inaccessible; if we succeed to get a disjoint system D of intervals of L such that $pD = dL$, then obviously D would be a requested system of maximal power because $cL \leq sL$. Now, for every chain L one has $cL \leq sL \leq c(L)^+$ (Theor. 2 p. 121 in 1935:2,3); therefore, since by hypothesis sL is inaccessible, we infer that necessarily $cL = sL$, thus cL is inaccessible; therefore, by $L(i)$, L contains a disjoint system D of intervals such that $pD = cL$ thus $pD = sL$ what we wanted to show.

2:3. LEMMA. $ReH \Rightarrow (L(i) \Rightarrow P_5(i))$.

Proof. Let L be a chain such that cL is inaccessible; we want to get, assuming $L(i)$ and ReH, a disjoint system D of intervals of L such that $pD = dL$. First case: $cL = dL$; then dL is inaccessible and the application of $L(i)$ yields a disjoint system D of power $dL = cL$ and thus $P_5(i)$ is holding.

Second case: $cL < dL$ i.e. $dL = c(L)^+$, dL is accessible and we are not allowed to apply $L(i)$. Now, a dyadic atomization of L would yield a tree $(T, \supset)$ of intervals of rank $\omega_{(dL)}$ which is not attained (because of $cL < dL$); thus $p_cT \leq cL$; therefore, ReH would imply $pT \leq p_sT \cdot p_cT \leq cL \cdot cL = cT$, contrarily that $pT = \aleph_{(dL)} > cL$.

3. Propositions P_0', $P_s(:= P_{18})$, P_s', P_s''

3:0. *Definition.* A tree (pseudotree) T is said to be branching (almost branching) if for every $x \in T$ one has $pR_0T(x,.) > 1$ $(p_sT(x,.) > 1)$, where

$$T(x,.) := \{y : y \in T \,\&\, x < y\};\;\; R_0A := \{a : a \in A :\; A(.,a) = \nu\}.$$

3:1. LEMMA. *If a tree T is branching, then to every chain L of T corresponds an equinumerous antichain $A(L)$ of T; $p_cT \leq p_sT$; T is ν (vacuous) or infinite. If $T \neq \nu$, then the numbers pT, p_cT, p_sT are infinite.*

Proof of 3:1. It is sufficient to associate with every $x \in L$ a point $x' \in RT_0(x,.)$ and to denote by $A(L)$ the set of all such point $x'(x \in L)$. Therefore $p_cT := \sup_L pL = \sup_L pA(L) \leq \sup pA$ (A running trough the system all antichains of T) $:= p_sT$. Of course, $p_c\nu = p_s\nu = p\nu = 0$. If $T \neq \nu$, then every $x \in T$ has some immediate successor fx; the set $L := \{x, fx, f^2x, \dots\}$ is an infinite chain in T.

In connection with P_0 (s. №. 0:1:0) let us formulate the following intriguing statement P_0'.

3:2. PROPOSITION P_0'. *Every branching tree is equinumerous to a free subset (any antichain is called also a free set).*

3:3. THEOREM. $P_0' \Leftrightarrow RP :\equiv P_2$ (s. № 0:2).

Proof. The implication $P_0' \Rightarrow RP$ is obvious, because every free subset is a special case of a degenerate set; therefore, let us prove the converse $RP \Rightarrow P_0'$. Now, let

T be any infinite branching tree; if $T = \nu$, all is obvious; if $T \neq \nu$, then T is infinite (s. L. 3:1): in virtue of RP, T contains a degenerate subset D of power pT. The first row R_0D of D is free. If incidentally, the antichain R_0D is of power $pD(= pT)$, all is done. If $pR_0D < pD$, then the set $E := D \setminus R_0D$ as well as the set $F := \cup T(.,x] \setminus \cup T(.,y]$ $(x \in E,\ y \in R_0D)$ are degenerate and of a power $< pD$. Since T is branching, every $f \in F$ has in T at least 2 immediate successors. Let sf be an immediate successor of f such that (i) $sf \notin F$; sf exists because F is degenerate. Let $sF := \{sF : f \in F\}$. Then sF is a requested free subset of T of power pT. As a matter of fact, first, $psF = pF$, because the mapping $s|F$ is one-to-one: if $f, g \in F$ and $sf = sg$, then $(sf)^- = (sg)^-$, i.e. $f = g$. Secondly, sF is free in T; in the opposite case, there would be 2 distinct points f, g in F such that the distinct points sf, sg would be comparable: either (ii) $sf < sg$ or (iii) $sf > sg$. But neither (ii) nor (iii) is holding. Assume (ii); then g as the immediate predecessor of sg would satisfy $sf \leq g \in F$, and consequently $sf \in F$, contrarily to (i). Analogously, one proves that (iii) is not possible. Q.E.D.

3:4. PROPOSITION P_s'. *If a tree T is branching, then the width p_sT is such that $p_sT = p_sT^2$.*

3:5. PROPOSITION P_s''. *If a tree T is branching, then the free power p_sT is attained.*

3:6. PROPOSITION $P_s := P_{18}$. *The width of every branching tree is attained and equals the width of the tree: $P_s := P_s' \,\&\, P_s''$.*

3:7. THEOREM. $RH \Leftrightarrow P_s'$.

Proof.

3:7:1. LEMMA. $P_s \Rightarrow RH$.

In the opposite case, there would be a non reflexive infinite tree T; there is no restriction to assume that pT be regular; then (s. 1935:2,3 p. 109, №. 11.3 Theor. 2) T would contain a distinguished subtree $A = A(T)$ ($\equiv$ Aronszajn subtree) of power pT. Let us consider $X := \cup R_0A(y,\cdot) \times R_0A(y,\cdot)$, $(y \in A(T))$; then X is an antichain in T^2. Therefore, (i) $p_sA^2 = pA = pT$. Since A is branching, P_s'' implies that p_sA is attained; by (i) this means that A is equinumerous to a proper subantichain; therefore also T is equinumerous to the same antichain: T would be D-reflexive, contrarily to the hypothesis.

3:7:2. LEMMA. $RH \Rightarrow P_s''$.

Since $RH \Rightarrow ReH$ the Lemma 3:7:2 is implied by the following.

3:7:3. LEMMA. *ReH implies that every branching tree T satisfies (i) $p_cT \leq p_sT$, $pT = p_sT$, $p_sT = p_sT^2$; p_sT is attained.*

Proof. Since T is branching, the first relation in (i) is holding (v. Lemma 3:1). On the other hand by ReH we have (ii) $p_cT,\ p_sT \leq pT \leq p_cT \cdot p_sT$. Since $p_cT \leq p_sT$ and for $T \neq \nu$ the numbers p_cT, p_sT are infinite, one has $p_cT \cdot p_sT = p_sT$ and (ii) yields (iii) $pT = p_sT$ thus, p_sT is determined.

What about the attainability of p_sT? If $p_sT := n$ is accessible, then this number n is attained (v. 1987:1 Theor. 2:4 with corresponding comments). Remains the case that n, thus by (iii) pT too, is inaccessible. Now, by RH, T contains a D-subset D of power pT; if for some $x \in D$ the chain $L := D[x,.)$ is of power pT, then $A(L)$ is an antichain of power pT, thus n is attained. If for every $x \in D$ the chain $D[x,.)$ is of a power $< pT$, then the disjoint partition $D[x,.)\,(x \in D)$ of D implies $pR_0D = pT$, because, by hypothesis, pT is regular; since R_0D is an antichain, the attainability of n is established. Q.E.D.

3:7:4. Lemma. $ReH \Rightarrow P'_s$.

Proof. In the opposite case, there would be a branching tree T such that (i) $p_sT < p_sT^2$. But, by Lemma 3:7:3, $p_sT = pT$ and n is attained; this means that T contains an antichain X of power pT and (i) would yield $pT < p_sT^2$; this inequality contradicts the relations $p_sT \le (pT^2) = pT$.

3:7:5. Lemma. $RH \Rightarrow P'_s$.

This follows from $RH \Rightarrow ReH$ and $ReH \Rightarrow P'_s$ (s. Lemma 3:7:4). Finnaly, 3:7:2, 3:7:5 imply $RH \Rightarrow P_s$; and this joint to 3:7:1 imply the theorem 3:7.

3:8. Theorem. $ReH \Leftrightarrow P'_s$.

3:8:1. Proof of $\Rightarrow$. ReH implies that every branching tree T satisfies $pT = p_sT \ge \aleph_0$. Now, $p_sT \le p_sT^2 \le (pT)^2 =$ (because pT is infinite) $= pT = p_sT$, thus $p_sT = p_sT^2$.

3:8:2. Proof of $P'_s \Rightarrow ReH$. If this implication were false, there would exist an infinite tree T such that (i) $pT > p_sT \cdot p_cT$; pT would be necessarily of the form $\aleph_{\alpha+1}$ (cf. 1935:2,3 p. 105, Theor. 1); T would contain an equinumerous distiguished subtree A (v. 1935:2,3 p. 105, Theor. 2); A is branching; therefore, by P'_s, $p_sA = p_sA^2$. Now, $p_sA^2 = pA$ because $\cup R_0(x,.) \times R_0(x,.)\,(x \in A)$ is an antichain in A^2 of power $pA = pT$. Hence $p_sA = pA$ and therefore $p_sT = pT$ and (since $p_cT \le p_sT$) $p_cT \cdot p_sT = p_sT = pT$, contradicting (i).

3:9. *Problem.* Does ReH (or RH) imply that p_sT is attained in every almost branching tree? (cf. 3:0, 3:7:3).

4. Main theorem.

The following 9 statements are pairwise equivalent:

4:0. Tree Alternative (TA): *If T is any infinite tree, then $pT = p_cT$ or $pT = p_sT$* (cf. 1969: 7, № 8:5).

4:1. Tree $b = p$ Statement. *If T is any infinite tree, then $bT = pT$* (cf. 1935:2,3 p. 112, № 6 and 1969:8, Theor. 8:8).

4:2. Tree proposition $b' = p$: *Every infinite tree T satisfies $b'T = pT$* (s. 1935:2,3 p. 112, № 6)

where $b'T := \sup pF$, F running through all non radial systems of directions in $(T, \le)$. A direction in $(T, \le)$ is defined as every $(a,b) \in T^2$ such that either $a =$

$b \,\&\, T(a,.) = \nu(:=$ vacuous or b is an immediate successor of a in $(T, \leq)$. Direction (a, b) is said to be non radial with direction (a', b') if and only if $(a, b) \neq (a', b')$ and either $a \| a'$ or a non $\| a' \,\&\, b$ non $\| b'$. A non radial system of directions is any system of directions which are pairwise non radial.

4:3. Tree Rectangle Hypothesis (ReH) (v. № 1:0).

4:4. Tree square b-Statement (TSb) $bT^2 = bT$ *for every infinite tree,*

4:5. Linear order Square Cellularity Density Statement (Lcd) or P_5^0 (v. 1:3; cf. 1935:2,3 p. 121. Theor. 2).

4:6. LSc (Linear Order Square cellularity) *Every linearly ordered dense set L satisfies* $cL^2 = cL$ (cf. 1950: 8, 1952: 8, 1953: 12, 1953: 12).

4:7. P_s'(cf. №. 3:0, 3:4).

4:8. Tree Star Width Statement (TSW) $sT = p_sT$ *for every tree* T (v. №. 1:1).

4:9. *Proof.* A proof of Theorem 4 is given in such way that following lemmas 4:10 – 4:24 are proved or quoted.

4:10. Lemma. 4:0 $\Leftrightarrow$ 4:1. Proof is obvious.

4:11. Lemma. 4:1 $\Rightarrow$ 4:2. (Implied by $bT \leq b'T \leq pT$; s. 1935:2,3 p. 110, L. 3).

4:12. Lemma. 4:2 $\Rightarrow$ 4:1. (s. 1935:2,3 p. 112, №. 6).

4:13. Lemma. 4:0 $\Rightarrow$ 4:3.

The implication is obvious if T is finite. If T is finite, then TA implies that at least one of the numbers p_cT, p_sT equals pT; therefore, their product is pT.

4:14. Lemma. 4:3 $\Rightarrow$ 4:0.

This is obvious, because if one of positive cardinals a, b, c is infinite and $a \leq b \leq c \leq ab$, then $b = c$.

4:16. Lemma. 4:0 $\Rightarrow$ 4:4.

Since T is infinite, $pT = (pT)^2$; TA implies $pT = \sup\{p_cT, p_sT\} \leq bT \leq bT^2 \leq (pT)^2 = pT$, thus $bT = bT^2$.

4:17. Lemma. 4:4 $\Rightarrow$ 4:0.

In the opposite case there would be infinite tree T such that (i) p_cT, $p_sT < pT$; then (cf. 1935:2,3 p. 109, Th. 2) T would be equinumerous to a distinguished subtree $A = A(T)$. For every $x \in A$ the set $fx \times fx\,(fx := R_0(x\cdot,)$ denotes the set of all immediate followers of x in A) is an infinite antichain in T^2; so is also the union $U := \cup(fx)^2 (x \in A)$. Since $pU = pA = pA = pT$, the square T^2 would contain the antichain U of power pT, thus $bT^2 = pT$, contrarily to R:4 and the assumption (i).

4:18. Lemma. 4:0 $\Rightarrow$ 4:5.

Proof. In the opposite case, there would exist an infinite linearly ordered set L such that (i) $cL < dL$. Let D be any dyadic atomization of L; then $pD = dL$,

$p_sD = cL$. In virtue of (i) one has (ii) $p_sD < pD$ and therefore, by 4:0, (iii) $pD = p_cD$. Now, if pD is of form $\aleph_{n+1}$, D would contain a chain of power $pD > p_sL$, if pD is a limit cardinal, then for every cardinal $n < pD$, and in particular for $n := (p_sD)^+ = (cL)^+$, there would be a subchain C in $(D, \supset)$ such that $pC = n$; but this is not possible, because the sets int $(X \setminus X^+)(X \in C, X^+$ is the immediate successor of X in $C)$ would form a system of power n of non empty open sets in L–absurdity.

4:19. LEMMA. 4:5 $\Rightarrow$ 4:0.

Proof. In the opposite case, there would exist an infinite tree T such that neither $pT = p_cT$ nor $pT = p_sT$; then T would contain a distinguished equinumerous subtree A (s. 1935:2,3 p. 109, Th. 2). Let N be any node of A and $(N, <_N)$ any total order of N having no first element; then the "natural ordering $(A, <_n)$ is a total order in A which is an extension of $<_A$ and of $<_N$ for every node N of $(A, <_A)$. But in this chain L one has $cL = p_sA$, $dL = pA$, thus $cL < dL$, contrarily to the assumption 4:5.

4:20. LEMMA. 4:0 $\Rightarrow$ 4:6.

Proof. For every infinite chain L we have (i) $dL = cL := n$ or (ii) $dL = n^+$ (s. 1935:2,3 p. 121, Th. 2). Since TA $\Rightarrow$ Lcd (v. L. 4:18) the case (ii) is excluded; thus (i) is holding. Now, $cL^2 \leq dL^2 = (dL)^2 = dL =$(by (i)) $= cL$, thus $cL^2 \leq cL$ and finally $cL^2 = cL$.

4:21. LEMMA. LSc $\Rightarrow$ TA, i.e. 4:6 $\Rightarrow$ 4:0.

A proof runs like the one in Lemma 4:19; the preceding chain $(A, <_n) := L$ would be such that $cL = p_sA < pL$ and therefore (i) $cL < pL$. Now, any dyadic atomization of L produces a tree $(D, \supset)$ of segments of L of power > 1 each and such that $pD = pL$. If for $X \in D$ one denotes by X_0, X_1 the two successors of X in $(D, \supset)$, then the non empty interiors of $X_0 \times X_1$ $(X \in D)$ would be pairwise disjoint open sets in L thus $cL^2 = pL$ and by (i) one would have $cL^2 > cL$, in contradiction with the assumption 4:6.

4:22. LEMMA. 4:0 $\Rightarrow$ 4:7 i.e. $TA \Rightarrow P'_s$.

This follows from $TA \Rightarrow ReH$ (s. L. 4:13) and $ReH \Rightarrow P'_s$ (s. L. 3:7).

4:23. LEMMA. $P'_s \Rightarrow TA$, i.e. 4:7 $\Rightarrow$ 4:0.

A proof is contained in the proof of Lemma 3:5:1.

4:24. LEMMA. $TA \Rightarrow$ TSW, i.e. 4:0 $\Rightarrow$ 4:8.

The holding of this statement is implied by $TA \Rightarrow ReH$ (s. L. 4:13) and by $ReH \Rightarrow$ TSW (s. №. 1:6).

4:25. *Remark.* In this section we were not worried about the question if cardinal numbers we considered were attained when each of them was defined as a supermum.

5. Propositions $P_{19}, P_{20}, \ldots, P_{45}$

5:0. Theorem. *If* $X \in \{4{:}0,\ 4{:}1,\ \ldots,\ R{:}8\} = \{TA,\ b = p,\ b' = p$, ReH, bTS, Lcd, LSc, P'_s, TSW$\}$ *and* $y \in \{L(1)$, MATH, $P_5(\mathrm{i})\}$*, then* $X \,\&\, Y \Leftrightarrow RH$.

In other words, if 5:1 $(P_{19},\ P_{20},\ \ldots,\ P_{45})$ is the lexicographical ordering of the conjuctions of terms of the Cartesian product

5:2. $\{4{:}0, 4{:}1, \ldots, 4{:}8\} \times \{L\,(\mathrm{i})$, MATH, $P_s\,(\mathrm{i})\}$,

then every term of 5:1 is equivalent to RH; in particular, the term $P'_5 \,\&\, P_5(\mathrm{i}) := P_{36}$ is equivalent to RH.

Proof. Since P_5, P'_{15}, ReH are pairwise equivalent (s. Theorem 2:1) and since ReH $\Rightarrow (L\,(\mathrm{i}) \Rightarrow P_5\,(\mathrm{i}))$ (s. Lemma 2:3) and MATH $\Leftrightarrow L\,(\mathrm{i})$ (s. 1:5) each of the propositions in 5:1 is equivalent to P_{36}. By Theorem 1:7 we have $P_5 \Leftrightarrow P_{36}$; since P_5 is equivalent to RH the theorem 5:0 is completly proved. Putting together the previous results we have the following.

5:3. Theorem. The 50 propositions $P_0, P_1, \ldots, P_{45}, \acute{P_0}, \acute{P_2}, P_b, \bar{P_{17}}$ are pairwise equivalent.

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