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APPROXIMATION OF CONTINUOUS FUNCTIONS BY MONOTONE SEQUENCES OF POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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Abstract. The problem of approximation by polynomials with restricted coefficients, is considered in several papers (e.g. [7–9]). In [1–6], I have proved among other things, that every $f \in C_{[0,1]}$ can be approximated uniformly by a polynomial sequence $(P_n)_n$ such that $(P_n)_n$ is monotonically decreasing on $[0,1]$. The aim of this paper is to extend the ideas of [1–6] to the case of approximation by polynomials with restricted coeficients.

1. Introduction

Let $C_0[0,1] = \{f \in C_{[0,1]};\ f(0) = 0\}$ and let $A = \{A_k\}_{k\geq 1}$ be a sequence of positive real numbers and

$$P_A = \left\{ p = \sum_{k=1}^{n} a^k x^k;\ n \in N,\ |a_k| \leq A_k^k,\ \text{ for all } \ k = 1, 2, \dots \right\}.$$

The problem of approximation in the space $C_0[0,1]$ by polynomials $P_n \in P_A$, is treated in several papers, (see e.g. [7–9]), where it is proved for example [9] that P_A is dense in $C_0[0,1]$ iff there is a subsequence of natural numbers $\{k_i\}_i$ satisfying the properties

$$\sum_{i=1}^{\infty} 1/k_i = +\infty \ \text{ and } \ \lim_{i\to\infty} A_{ki} = +\infty.$$

In several papers ([1–6]) I have proved, among other things, that every $f = C_{[0,1]}$ can be approximated uniformly by a polynomial sequence $(P_n)_n$, monotonically decreasing on $[0,1]$.

In the present paper we will extend the ideas of [1–6] to the case of approximation in $C_0[0,1]$ by polynomials from P_A.

2. Basic Result

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Let us consider $\{A_k\}_{k\geq 1}$ which satisfies:

$$1 \leq A_1 \leq A_2 \leq \cdots \leq A_k \leq A_{k+1} \leq \ldots, \lim A_i = +\infty. \tag{1}$$

Then, by [9], P_A is dense in $C_0[0,1]$. Now, let us denote by

$$C_0^1[0,1] = \{f \in C_0[0,1]: \ f'(0) = 0\}.$$

THEOREM 2.1. *For every $f \in C_0^1[0,1]$, there exists a polynomial sequence $P_n \in P_A$, $n = 1, 2, \ldots$, such that $P_n \to f$ uniformly on $[0,1]$ and $f(0) = P_n(0)$, $f(x) < P_{n+1}(x) < P_n(x)$, for all $x \in (0,1]$ and all $n = 1, 2, \ldots$.*

Proof. Take $F(x) = f(x)/x$, $x \in (0,1]$, $F(0) = 0$. Since $f \in C_0^1[0,1]$ it is evident that $F \in C_0[0,1]$. Then, P_A is dense in $C_0[0,1]$ and therefore there exists a polynomial sequence $(R_n)_n$, $R_n \in P_A$, $n = 1, 2, \ldots$, such that $|F(x) - R_n(x)| < 1/[n(n+1)]$, for all $x \in [0,1]$, and $n = 1, 2, \ldots$. Hence:

$$|f(x) - xR_n(x)| \leq x/[n(n+1)], \text{ for all } x \in [0,1], \text{ and } n = 1, 2, \ldots. \tag{2}$$

Take $Q_n(x) = xR_n(x)$ and $S_n(x) = Q_n(x) + 2x/n$, $x \in [0,1]$, $n = 1, 2, \ldots$. From (2) it is evident that $Q_n \to f$ uniformly and therefore $S_n \to f$ uniformly on $[0,1]$. Then by (2) we obtain:

$$\begin{aligned} |Q_n(x) - Q_{n+1}(x)| &\leq |Q_n(x) - f(x)| + |f(x) - Q_{n+1}(x)| \leq \\ &\leq x/[n(n+1)] + x/[(n+1)(n+2)] \leq 2x/[n(n+1)], \end{aligned}$$

for all $x \in [0,1]$ and all $n = 1, 2, \ldots$, and therefore

$$S_n(x) - S_{n+1}(x) = Q_n(x) - Q_{n+1}(x) + 2x/[n(n+1)] > 0$$

for all $x \in (0,1]$ and $S_n(0) = S_{n+1}(0) = f(0)$ for all $n = 1, 2, \ldots$.

Now, let $n \in N$ be fixed and $R_n(x) = \sum_{k=1}^{j_n} a_k x^k \in P_A$; therefore $|a_k| \leq A_k^k$, for all $k = \overline{1, j_n}$. Then $xR_n(x) = \sum_{k=1}^{j_n} a_k x^{k+1} = \sum_{i=2}^{j_n+1} a_{i-1} x^i$. We have:

$$|a_{i-1}| \leq A_{i-1}^{i-1} \leq A_i^{i-1} \leq A_i^i, \ i = \overline{2, j_n + 1};$$

therefore, it is evident that $xR_n(x) = Q_n(x) \in P_A$, $n = 1, 2, \ldots$. But $S_n(x) = \frac{2}{n}x + \sum_{i=2}^{j_n+1} a_{i-1} x^i$ and it is evident $2/n \leq A_1$ for all $n \geq 2$. Hence $S_n \in P_A$ for all $n \geq 2$ and therefore it is evident that $P_n(x) = S_{n+1}(x)$, $n = 1, 2, \ldots$, satisfies the conclusion of Theorem 2.1.

Remark. If in the previous proof we consider $S_n(x) = Q_n(x) - 2x/n$, then it is easily seen that $(S_n)_n$ is monotonically increasing on $(0,1]$, $n = 2, 3, \ldots$.

3. Some Extensions

Let us consider $m, r_k \in N$, $k = \overline{1, m}$ and the numbers $D_i^{(k)} \in R$, $i = \overline{0, r_k}$, $k = \overline{1, m}$ satisfying

$$\sum_{i=0}^{r_k} D_i^{(k)} = 0, \ \sum_{i=0}^{r_k} |D_i^{(k)}| > 0 \text{ for all } k = \overline{1, m}. \tag{3}$$

Also, let $A = \{A_k\}_{k\geq 1}$ satisfy (1). We will prove result analogous to Thereom 1.2 of [4]:

THEOREM 3.1. *If there is a sequence $(\alpha_n)_n$, $\alpha_n \in R$, $\alpha_n \searrow 0$ satisfying*

$$\sum_{i=0}^{r_k} D_i^{(k)} \alpha_{n+r_k-i} > 0 \text{ for all } k = \overline{1,m} \text{ and } n \in N \tag{4}$$

then for every $f \in C_0^1[0,1]$, there exists a polynomial sequence $P_n \in P_A$, $n = 1, 2, \ldots$, such that $P_n \to f$ uniformly on $[0,1]$ and

$$\begin{gathered} \sum_{i=0}^{r_k} D_i^{(k)} P_{n+r_k-i}(x) > 0, \textit{ for all } x \in (0,1],\ k = \overline{1,m},\ n \in N \\ P_n(0) = f(0), \textit{ for all } n = 1, 2, \ldots \end{gathered} \tag{5}$$

Proof. By Lemma 1.1 of [4] we can write:

$$\sum_{i=0}^{r_k} D_i^{(k)} \alpha_{n+r_k-i} = \sum_{i=0}^{r_k-1} C_i^{(k)} (\alpha_{n+r_k-i} - \alpha_{n+r_k-i-1})$$

$$\text{with } C_i^{(k)} = \sum_{j=0}^{i} D_j^{(k)},\ i = \overline{1, r_k - 1},\ k = \overline{1,m}.$$

Then, as in the proof of Theorem 1.2 of [4], take:

$$a_n = \min\left\{ \sum_{i=0}^{r_k-1} C_i^{(k)} (\alpha_{n+r_k-i} - \alpha_{n+r_k-i-1});\ k = \overline{1,m} \right\},$$

$$D = \max\left\{ \sum_{i=0}^{r_k-1} |C_i^{(k)}|;\ k = \overline{1,m} \right\} > 0,\ \gamma_m = a_n/(2D),\ n \in N$$

and let us consider $\beta_n < \gamma_n$, $n \in N$, $\beta_n \searrow 0$. Now, for $F(x) = f(x)/x$, $x \in (0,1]$, $F(0) = 0$, $F \in C_0[0,1]$, there exists a polynomial $(R_n)_n$, $R_n \in P_A$ such that $|F(x) - R_n(x)| < \beta_n$, for all $x \in [0,1]$, and all $n = 1, 2, \ldots$. Hence $|f(x) - xR_n(x)| \leq x\beta_n$, for all $x \in [0,1]$ and all $n = 1, 2, \ldots$.

Take $Q_n(x) = xR_n(x)$ and $P_n(x) = Q_n(x) + x\alpha_n$. Then, reasoning exactly as in the proof of Theorem 1.2 of [4], (5) follows immediately. Also, as in the proof of Theorem 2.1 (because there is an $n_0 \in N$ such that $\alpha_n \leq A_1$, for all $n \geq n_0$), $P_n \in P$, for all $n \geq n_0$ and therefore the sequence $P_{n+n_0}(x)$, $n = 1, 2, \ldots$, satisfies the conclusion of Theorem 3.1.

As in immediate corollary (analogous to Corollary 1.3 of [4]) we obtain the following:

COROLLARY 3.2. *For any $f \in C_0^1[0,1]$ and any $r_0 \in N$, there exists a polynomial sequence $P_n \in P_A$, $n = 1, 2, \ldots$, uniformly convergent to f on $[0,1]$*

which satisfies

$$(1-)^{r_k}\sum_{i=0}^{r_k}(-1)^i\binom{r_k}{i}P_{n+r_k-i}(x)>0,\ \textit{for all}\ x\in(0,1],\ n\in N,$$

$$r_k\in N,\ r_k\le r_0,\ k=\overline{1,m}\quad\textit{and}\ P_n(0)=f(0)\ \textit{for all}\ n=1,2,\ldots$$

Remark. (1) If $r_0=2$, from the previous inequality we obtain that $P_{n+1}(x)<P(x)$ and $P_{n+2}(x)-2P_{n+1}(x)+P_n(x)>0$, (namely $(P_n)_n$ is "convex" on $[0,1]$), and $P_n(0)=f(0)$ for all $n=1,2,\ldots$.

Added in proof. (2) These results do not remain valid for all $f\in C_0[0,1]$. One such example is the following. Let us denote $D(f;x)=[f(x)-f(0)]/x$, $(D^*f)(0)=\limsup_{x\downarrow 1}D(f;x)$ and let $f:[0,1]\to R$ be defined by $f(0)=0$, $f(x)=-1/\ln x$ if $x\in(0,1/2]$ and $f(x)=1/\ln 2$ if $x\in(1/2,1]$, where $\ln(x)$ represents the hyperbolic logarithm of x. We have: $\lim_{x\downarrow 0}f(x)=-1/-\infty=0$ and $\lim_{x\uparrow 1/2}f(x)=1/\ln 2$ wich implies $f\in C_0[0,1]$, but f does not satisfy Theorem 2.1.

Indeed, if Theorem 2.1 held for the function f so defined, it would follow that there exists a sequence $P_n\in P_A$, $n\in N$ such that $P_n\to f$ uniformly on $[0,1]$ and $0=f(0)=P_n(0)$, $f(x)<P_{n+1}(x)<P_n(x)$, $\forall x\in(0,1]$, $\forall n\in N$. Hence we would obtain $D(f;x)=f(x)/x<P_n(x)/x=D(P_n;x)$ $\forall x\in(0,1]$, therefore $(D^*f)(0)\le(D^+P_n)(0)=P_n'(0)$, $n=1,2,\ldots$. But $D(f;x)=-1/[x\cdot\ln x]$ $\forall x\in(0,1/2)$ which immediately implies $(D^+f)(0)=+\infty$ choosing for example $x_m\searrow 0$, $x_m=e^{-m}$, $m\in N$) contradicting the inequalities $(D^+f)(0)\le P_n'(0)$, $n\in N$.

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