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ASYMPTOTIC BEHAVIOR OF PARTIAL SUMS OF FOURIER-LEGENDRE SERIES

R. Bojanić and **Z. Divis**

If f is defined and has a derivative of bounded variation on $[-1,1]$ the main result of this paper is the asymptotic formula for the partial sums of the Fourier-Legendre expansion of f:

$$S_n(f,x) = f(x) + (n\pi)^{-1}\sqrt{1-x^2}(f'_R(x) - f'_L(x)) + o(1/n).$$

Here $f'_R(x)$ and $f'_L(x)$ are the right and the left derivatives of f at $x \in (-1,1)$.

1. Let $P_n(x)$ be the Legendre polynomial of degree n normalized so that $P_n(1) = 1$. We shall study here the asymptotic behavior of the partial sums of the Fourier-Legendre series of a function f whose derivative is of bounded variation on $[-1,1]$.

We shall assume here that

$$f(x) = f(-1) + \int_{-1}^{x} \varphi(t)\,dt,\ x \in (-1,1) \tag{1.1}$$

where φ is a function of bounded variation on $[-1,1]$. It is clear that the left and right derivatives

$$f'_L(x) = \varphi(x-0) \ \text{ and } \ f'_R(x) = \varphi(x+0)$$

exists at every point $x \in (-1,1)$.

The Fourier-Legendre series of f is the series

$$\sum a_k(f)P_k(x)(k \geq 0) \quad \text{where} \quad a_k(f) = \left(k + \frac{1}{2}\right)\int_{-1}^{1} f(t)P_k(t)\,dt.$$

Since f is clearly a continuous function of bounded variation, the Fourier-Legendre series of f converges to $f(x)$ at every $x \in (-1,1)$ (see [1], [2], [3]). An estimate for the rate of convergence of the partial sums

$$S_n(f,x) = \sum_{k=0}^{n} a_k(f)P_k(x)$$

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to $f(x)$ in this case is given by

$$|S_n(f,x) - f(x)| \le \frac{M(x)}{n} \sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(f)$$

(see [4]). Here $V_a^b(f)$ denotes the total variation of f on $[a,b]$.

The main purpose of this paper is to show that for functions f satisfying condition (1.1) it is possible to obtain the next term of the asymptotic expansion of $S_n(f,x)$ as $n \to \infty$.

Our result, in its simplest form, can be can be stated as follows:

$$S_n(f,x) = f(x) + \frac{(1-x^2)^{1/2}}{n\pi}(f_R'(x) - f_L'(x)) + R_n(f,x) \tag{1.2}$$

where $R_n(f,x) = o(1/n)$ $(n \to \infty)$.

A more precise estimate of the remainder term $R_n(f,x)$ is the main result of this paper.

Theorem 1. *Suppose that the function f satisfies condition (1.1) for all $x \in [-1,1]$. We have then, for all $x \in (-1,1)$,*

$$\begin{aligned}(1-x^2)^{1/2}|R_n(f,x)| \le Cn^{-3/2} \sum_{k=1}^{n} k^{-1/2} V_{x-(1+x)/k}^{x+(1-x)/k}(\varphi_x) \\ + Cn^{-2}|f_R'(x) - f_L'(x)|.\end{aligned} \tag{1.3}$$

If

$$\int_{-1}^{1} (1-t^2)^{-1/4} dV_{-1}^{t}(\varphi_x) < \infty \tag{1.4}$$

then

$$\begin{aligned}(1-x^2)|R_n(f,x)| \le Cn^{-2} \sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(\varphi_x) \\ + Cn^{-2}\left(\int_{-1}^{1}(1-t^2)^{-1/4} dV_{-1}^{t}(\varphi_x) + |f_R'(x) - f_L'(x)|\right).\end{aligned} \tag{1.5}$$

Here

$$\varphi_x(t) = \begin{cases} \varphi(t) - \varphi(x-0) & \text{if } -1 \le t < x; \\ 0 & \text{if } t = x; \\ \varphi(t) - \varphi(x+0) & \text{if } x < t \le 1. \end{cases}$$

Note that $\varphi_x(t)$ is a function of bounded variation on $[-1,1]$, continuous at $t = x$, with $\varphi_x(x) = 0$. This clearly implies that $R_n(f,x) = o(1/n)(n \to \infty)$.

If f' is a continuous function of bounded variation, we have the inequality

$$(1-x^2)^{1/2}|S_n(f,x) - f(x)| \le Cn^{-3/2} \sum_{k=1}^{n} k^{-1/2} V_{x-(1+x)/k}^{x+(1-x)/k}(f')$$

or

$$(1-x^2)|S_n(f,x)-f(x)| \le Cn^{-2}\sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(f') + Cn^{-2}\int_{-1}^{1}(1-t^2)^{-1/4}dV_{-1}^{t}(f)$$

if the last integral is finite.

More general results of this type for the asymptotic behavior of partial sums of the Fourier series of a 2π-periodic function whose r-th derivative ($r \ge 1$) is of bounded variation, were obtained recently by Z. Divis [5].

To see that the estimate (1.5) cannot be improved it is sufficient to consider the function $f(x) = |x|$ at the point $x = 0$. We have in this case

$$\varphi(t) = \begin{cases} -1, & -1 \le t < 0 \\ 1, & 0 < t \le 1 \end{cases}$$

so that $\varphi_0(t) = 0$ for all $t \in [-1,1]$. From (1.2) and (1.3), with n replaced by $2n$, we find that $|S_{2n}(f,0) - 1/(n\pi)| \le C/(2n^2)$.

On the other hand, it is known that the Fourier-Legendre expansion of the function $f(x) = |x|$ is

$$|x| = \frac{1}{2} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{4k+1}{(2k-1)(k+1)}P_{2k}(0)P_{2k}(x).$$

Thus, at $x = 0$,

$$S_{2n}(f,0) = \frac{1}{2}\sum_{k=n+1}^{\infty}\frac{4k+1}{(2k-1)(k+1)}P_{2k}^2(0).$$

Since

$$P_k(\cos\theta) = \left(\frac{2}{\pi k\ \sin\theta}\right)^{1/2}\cos\left(\left(k+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right) + O(k^{-3/2})$$

we have, for $\theta = \pi/2$,

$$P_{2k}(0) = (1/\pi k)^{1/2}\cos(k\pi) + O(k^{-3/2}).$$

Hence

$$S_{2n}(f,0) = \frac{1}{2}\sum_{k=n+1}^{\infty}\frac{4k+1}{(2k-1)(k+1)}\left(\frac{1}{\pi k} + O(k^{-2})\right) = \frac{1}{\pi}\sum_{k=n+1}^{\infty}\frac{1}{k^2} + O\left(\frac{1}{n^2}\right)$$

or

$$S_{2n}(f,0) = 1/(\pi n) + O(1/n^2).$$

2. The proof of Theorem 1 it based on a transformation of the formula

$$S_n(f,x) - f(x) = \int_{-1}^{1}(f(t)-f(x))K_n(x,t)\,dt$$

where

$$K_n(x,t) = \sum_{k=0}^{n} (k+1/2) P_k(x) P_k(t), \tag{2.1}$$

by means of partial integration. We have, for $x \in (-1,1)$

$$S_n(f,x) - f(x) = \left(\int_{-1}^{x} + \int_{x}^{1} \right) (f(t) - f(x)) K_n(x,t)\, dt.$$

If $A_n(x,t) = \int_{-1}^{t} K_n(x,u)\, du$ and $B_n(x,t) = \int_{t}^{1} K_n(x,u)\, du$ we have, by (1.1) and partial integration,

$$\begin{aligned}\int_{-1}^{x} (f(t) - f(x)) K_n(x,t)\, dt &= -\int_{-1}^{x} \varphi(t) A_n(x,t)\, dt \\ &= -\varphi(x-0) \int_{-1}^{x} A_n(x,t)\, dt - \int_{-1}^{x} (\varphi(t) - \varphi(x-0)) A_n(x,t)\, dt\end{aligned}$$

and

$$\begin{aligned}\int_{x}^{1} (f(t) - f(x)) K_n(x,t)\, dt &= \int_{x}^{1} \varphi(t) B_n(x,t)\, dt \\ &= \varphi(x+0) \int_{x}^{1} B_n(x,t)\, dt + \int_{x}^{1} (\varphi(t) - \varphi(x+0)) B_n(x,t)\, dt.\end{aligned}$$

Thus

$$\begin{aligned}S_n(f,x) - f(x) = f'_R(x) \int_{x}^{1} B_n(x,t)\, dt - f'_L(x) \int_{-1}^{x} A_n(x,t)\, dt \\ - \int_{-1}^{x} (\varphi(t) - \varphi(x-0)) A_n(x,t)\, dt + \int_{x}^{1} (\varphi(t) - \varphi(x+0)) B_n(x,t)\, dt.\end{aligned}$$

This formula can be further simplified by observing that

$$\int_{-1}^{x} A_n(x,t)\, dt = -\int_{-1}^{x} (t-x) K_n(x,t)\, dt = \int_{x}^{1} (t-x) K_n(x,t)\, dt$$

and

$$\int_{x}^{1} B_n(x,t)\, dt = \int_{x}^{1} (t-x) K_n(x,t)\, dt.$$

We have used here the fact that, for $n \geq 1$,

$$\int_{-1}^{1} (x-t) K_n(x,t)\, dt = \frac{n+2}{2} \int_{-1}^{1} (P_{n+1}(x) P_n(t) - P_n(x) P_{n+1}(t))\, dt = 0.$$

Thus the starting point of our investigations is the formula

$$\begin{aligned}S_n(f,x) - f(x) &= (f'_R(x) - f'_L(x)) \int_{x}^{1} (t-x) K_n(x,t)\, dt \\ &\quad - \int_{-1}^{x} \varphi_x(t) A_n(x,t)\, dt + \int_{x}^{1} \varphi_x(t) B_n(x,t)\, dt.\end{aligned} \tag{2.2}$$

The proof of Theorem 1, in view of (2.2), is a consequence of the following results.

THEOREM 2. *If $K_n(x,t)$ is the Legendre kernel defined by* (2.1) *then for $-1 < x < 1$ we have*

$$\left|\int_x^1 (t-x)K_n(x,t)\,dt - (1-x^2)^{1/2}/(n\pi)\right| \le C(1-x^2)^{-1/2}n^{-2}. \tag{2.3}$$

In the following we will assume that $g(t)$ is a function of bounded variation on $[-1,1]$, with $g(x)=0$, for x fixed in $(-1,1)$.

THEOREM 3A. *Let $A_n(x,t) = \int_{-1}^t K_n(x,\nu)\,d\nu$. We have then*

$$(1-x^2)^{1/2}\left|\int_{-1}^x g(t)A_n(x,t)\,dt\right| \le Cn^{-3/2}\sum_{k=1}^n k^{-1/2}V_{x-(1+x)/k}^x(g). \tag{2.4A}$$

If $\left|\int_{-1}^x (1-t^2)^{-1/4}dV_t^x(g)\right| < \infty$ *then*

$$\begin{aligned}(1-x^2)&\left|\int_{-1}^x g(t)A_n(x,t)\,dt\right| \le \\ &\le Cn^{-2}\sum_{k=1}^n\left(V_{x-(1+x)/k}^x(g) + \left|\int_{-1}^x (1-t^2)^{-1/4}dV_t^x(g)\right|\right).\end{aligned} \tag{2.5A}$$

Likewise we have

THEOREM 3B. *Let $B_n(x,t) = \int_t^1 K_n(x,\nu)\,d\nu$. We have then*

$$(1-x^2)^{1/2}\left|\int_x^1 g(t)B_n(x,t)\,dt\right| \le Cn^{-3/2}\sum_{k=1}^n k^{-1/2}V_x^{x+(1-x)/k}(g). \tag{2.4B}$$

If $\int_x^1 (1-t^2)^{-1/4}dV_x^t(g) < \infty$ *then*

$$\begin{aligned}(1-x^2)&\left|\int_x^1 g(t)B_n(x,t)\,dt\right| \le \\ &\le Cn^{-2}\left(\sum_{k=1}^n V_x^{x+(1-x)/k}(g) + \int_x^1 (1-t^2)^{-1/4}dV_x^t(g)\right).\end{aligned} \tag{2.5B}$$

Theorems 3A and 3B are quite similar and it will be sufficient to prove only one of them.

It Section 3 we shall establish properties of Legendre polynomials necessary for the proofs of Theorems 2 and 3 and we shall prove Theorem 2. Finally, in Section 4 we shall give a proof of Theorem 3B.

3. The proofs of Theorems 2 and 3B are based on properties of the functions

$$B_n(x,t) = \int_t^1 K_n(x,\nu)\,d\nu \tag{3.1}$$

and

$$L_n(x,t) = \int_t^1 B_n(x,\nu)\,d\nu = \int_t^1 (\nu - t)K_n(x,\nu)\,d\nu \tag{3.2}$$

where $-1 \le x \le t \le 1$ and

$$K_n(x,t) = \sum_{k=0}^{n} (k+1/2)P_k(x)P_k(t)$$

is the Legendre kernel.

The functions $B_n(x,t)$ and $L_n(x,t)$ are clearly the n-th partial sums of the Fourier-Legendre expansions of the functions

$$\chi_t(\nu) = \begin{cases} 0 & \text{if } \nu < t \\ 1/2 & \text{if } \nu = t \\ 1 & \text{if } \nu > t \end{cases} \quad \text{and} \quad \psi_t(\nu) = \begin{cases} 0 & \text{if } \nu \le t \\ \nu - t & \text{if } \nu > t. \end{cases}$$

evaluated at $\nu = \chi$. Clearly,

$$\psi_t(\nu) = \int_{-1}^{\nu} \chi_t(u)\,du.$$

We have

$$B_n(\nu,t) = \int_{-1}^{1} \chi_t(u)K_n(\nu,u)\,du = \sum_{k=0}^{n} a_k(\chi_t)P_k(\nu)$$

where $a_k(\chi_t) = (k+1/2)\int_1^1 \chi_t(u)P_k(u)\,du$ or $2a_k(\chi_t) = (2k+1)\int_t^1 P_k(u)\,du$.

If $k=0$ we find that $a_0(\chi_t) = (1-t)/2$. For $k \ge 1$, using formula

$$(2k+1)\int_t^1 P_k(u)\,du = P_{k-1}(t) - P_{k+1}(t) \tag{3.3}$$

we find that $a_k(\chi_t) = (P_{k-1}(t) - P_{k+1}(t))/2$. Since the Fournier-Legendre series of the function $\chi_t(x)$ converges to $\chi_t(x)$, we have

$$\chi_t(x) = \sum_{k=0}^{\infty} a_k(\chi_t)P_k(x).$$

Now, if $x < t$ we have $\chi_t(x) = 0$. Consequently

$$\chi_t(x) = 0 = B_n(x,t) + \sum_{k=n+1}^{\infty} a_k(\chi_t)P_k(x)$$

or

$$B_n(x,t) = -\frac{1}{2}\sum_{k=n+1}^{\infty}(P_{k-1}(t) - P_{k+1}(t))P_k(x).$$

A partial summation gives

$$\begin{aligned} B_n(x,t) = &-\frac{1}{2}P_{n+1}(x)P_n(t) - \frac{1}{2}P_{n+2}(x)P_{n+1}(t) \\ &-\frac{1}{2}\sum_{k=n+2}^{\infty}(P_{k+1}(x) - P_{k-1}(x))P_k(t). \end{aligned} \tag{3.4}$$

Integrating this we find that for $x < t$

$$\begin{aligned} L_n(x,t) = &-\frac{1}{2}P_{n+1}(x)\int_t^1 P_n(\nu)\,d\nu - \frac{1}{2}P_{n+2}(x)\int_t^1 P_{n+1}(\nu)\,d\nu \\ &-\frac{1}{2}\sum_{k=n+2}^{\infty}(P_{k+1}(x) - P_{k-1}(x))\int_t^1 P_k(\nu)\,d\nu. \end{aligned}$$

Using relation (3.3) again, we find that

$$\begin{aligned} L_n(x,t) = &-\frac{1}{2}P_{n+1}(x)\int_t^1 P_n(\nu)\,d\nu - \frac{1}{2}P_{n+2}(x)\int_t^1 P_{n+1}(\nu)\,d\nu \\ &+\frac{1}{2}\sum_{k=n+2}^{\infty}\frac{(P_{k+1}(x) - P_{k-1}(x))(P_{k+1}(t) - P_{k-1}(t))}{2k+1}. \end{aligned} \tag{3.5}$$

Formulae (3.4) and (3.5) will be used to obtain inequalities for functions $B_n(x,t)$ and $L_n(x,t)$ $(-1 < x < t \le 1)$ needed in the proof of Theorem 3B.

If, in (3.5), we let $t \to x$ we find that

$$\begin{aligned} L_n(x,x) = &-\frac{1}{2}P_{n+1}(x)\int_x^1 P_n(\nu)\,d\nu - \frac{1}{2}P_{n+2}(x)\int_x^1 P_{n+1}(\nu)\,d\nu \\ &+\frac{1}{2}\sum_{k=n+2}^{\infty}\frac{(P_{k-1}(x) - P_{k+1}(x))^2}{2k+1}. \end{aligned} \tag{3.6}$$

This formula will be used in the proof of Theorem 2 for the evaluation of the asymptotic behavior of the function

$$L_n(x,x) = \int_x^1 (\nu - x)K_n(x,\nu)\,d\nu.$$

The following properties of Legendre polynomials will be used in the remaining part of this section:

$$|P_n(x)| \le (2/\pi)^{1/2}(1-x^2)^{-1/4}n^{-1/2} \tag{3.7}$$

(see [6, p. 163]);

$$\left|\int_x^1 P_n(t)\,dt\right| \le 2\sqrt{\pi}/n^{3/2} \quad (n \ge 2) \tag{3.8}$$

(see [7, p. 72]);

$$P_n(\cos\theta) = \left(\frac{2}{\pi n\,\sin\theta}\right)^{1/2} \cos\left(\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right) + O\left(\frac{1}{(n\,\sin\theta)^{3/2}}\right) \tag{3.9}$$

(see [6, p. 192]).

Another useful inequality, which is not a consequence of (3.9), is

$$|P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)| \le A(n^{-1}\sin\theta)^{1/2} \tag{3.10}$$

(see [8, p. 360]).

For the proof of Theorem 2 and 3B we need a more precise asymptotic formula for $P_{n-1}(x) - P_{n+1}(x)$.

Lemma 1. *For $0 \le \theta \le \pi$ we have*

$$P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta) = 2(\theta\,\sin\theta)^{1/2} J_1((n+1/2)\theta) + O((n^{-1}\sin\theta)^{3/2}) \tag{3.11}$$

and for $0 < \theta < \pi$,

$$\begin{aligned} P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta) = 2\left(\frac{2\,\sin\theta}{\pi n}\right)^{1/2} \sin\left(\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right) \\ + O\left(\frac{1}{(\sin\theta)^{1/2} n^{3/2}}\right). \end{aligned} \tag{3.12}$$

Proof of Lemma 1. Since Legendre polynomials are either even or odd, it is sufficient to consider only the case $0 \le \theta \le \pi/2$. We have

$$P_{n-1}(x) - P_{n+1}(x) = \frac{2n+1}{n(n+1)}(1-x^2)P_n'(x)$$

(see [8], p. 361). Since $P_n'(x) = 1/2 \cdot (n+1)P_{n-1}^{(1,1)}(x)$, we have

$$P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta) = (2n) + 1/(2n) \cdot \sin^2\theta P_{n-1}^{(1,1)}(\cos\theta).$$

But for $\alpha > -1$ and $0 < \theta \le \pi/2$ we have

$$\begin{aligned} \left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta} P_n^{(\alpha,\beta)}(\cos\theta) = N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_\alpha(N\theta) + \\ + O((\sin\theta)^{1/2} n^{-3/2}), \end{aligned}$$

where $N = n + (\alpha+\beta+1)/2$ and $J_\alpha(x)$ is the Bessel function of order α (see [6, p. 195 and the remark after Theorem 8.21.12]). For $\alpha = 1$, $\beta = 1$ we obtain, in particular

$$\sin\theta P_{n-1}^{(1,1)}(\cos\theta) = \frac{2n}{2n+1}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_1\left(\left(n+\frac{1}{2}\right)\theta\right) + O((\sin\theta)^{1/2}/n^{3/2})$$

and (3.11) follows.

Relation (3.12) is a consequence of the asymptotic relation

$$J_\alpha(x) = (2/\pi x)^{1/2}\cos(x - \alpha\pi/2 - \pi/4) + O(x^{-3/2}) \quad (x \to \infty)$$

(see [6, p. 15]). We have

$$\begin{aligned}
&P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)\\
&= 2(\theta\,\sin\theta)^{1/2}\biggl(\Bigl(\frac{2}{\pi(2n+1)\theta}\Bigr)^{1/2}\sin\Bigl(\Bigl(n+\frac{1}{2}\Bigr)\theta - \frac{\pi}{4}\Bigr) + O\Bigl(\frac{1}{(n\,\sin\theta)^{3/2}}\Bigr)\\
&+ O\Bigl(\Bigl(\frac{\sin\theta}{n}\Bigr)^{3/2}\Bigr) = 2\Bigl(\frac{2\,\sin\theta}{\pi n}\Bigr)^{1/2}\sin\Bigl(\Bigl(n+\frac{1}{2}\Bigr)\theta - \frac{\pi}{4}\Bigr) + O\Bigl(\frac{1}{(\sin\theta)^{1/2}n^{3/2}}\Bigr)\biggr)
\end{aligned}$$

and the proof of Lemma 1 is completed.

Proof of Theorem 2. From (3.6), (3.7) and (3.8) follows that

$$L_n(x,x) = \frac{1}{2}\sum_{k=n+1}^{\infty}\frac{(P_{k-1}(x) - P_{k+1}(x))^2}{2k+1} + O((1-x^2)^{-1/4}n^{-2}).$$

By Lemma 1 we have

$$\begin{aligned}
&(P_{k-1}(\cos\theta) - P_{k+1}(\cos\theta))^2\\
&\quad= \frac{8\,\sin\theta}{\pi k}\sin^2\Bigl(\Bigl(k+\frac{1}{2}\Bigr)\theta - \frac{\pi}{4}\Bigr) + O\Bigl(\frac{1}{(\sin\theta)k^2}\Bigr)\\
&\quad= \frac{4\,\sin\theta}{\pi k} - \frac{4\,\sin\theta}{\pi k}\cos\Bigl(\Bigl(2k+1\Bigr)\theta - \frac{\pi}{2}\Bigr) + O\Bigl(\frac{1}{(\sin\theta)k^2}\Bigr).
\end{aligned}$$

Hence, with $x = \cos\theta$,

$$\begin{aligned}
L_n(x,x) = \frac{2\,\sin\theta}{\pi}&\sum_{k=n+1}^{\infty}\frac{1}{(2k+1)k} - \frac{2\,\sin\theta}{\pi}\sum_{k=n+1}^{\infty}\frac{\sin((2k+1)\theta)}{(2k+1)k}\\
&+ O\Bigl(\frac{1}{(\sin\theta)n^2}\Bigr) + O\Bigl(\frac{1}{(\sin\theta)^{1/2}n^2}\Bigr)
\end{aligned}$$

or

$$L_n(x,x) = (1-x^2)^{1/2}/(\pi n) + O((1-x^2)^{-1/4}n^{-2})$$

and Theorem 2 is proved.

We shall use now formulae (3.4) and (3.5) to obtain estimates for functions $B_n(x,t)$ and $L_n(x,t)$ when $x < t$.

Lemma 2. *We have for* $-1 < x < 1$,

$$\int_x^1 B_n^2(x,\nu)\,d\nu \le C/n. \tag{3.13}$$

Proof of Lemma 2. From (3.4), which is valid for $t \geq x$, and the orthogonality relations for Legendre polynomials we find that

$$\begin{aligned}
&4\int_x^1 B_n^2(x,\nu)\,d\nu \\
&\leq \int_{-1}^1 \Big(P_{n+1}(x)P_n(\nu) + P_{n+2}(x)P_{n+1}(\nu) + \sum_{k=n+2}^{\infty}(P_{k+1}(x) - P_{k-1}(x))P_k(\nu)\Big)^2 d\nu \\
&\leq \frac{2}{2n+1}P_{n+1}^2(x) + \frac{2}{2n+3}P_{n+2}^2(x) + 2\sum_{k=n+2}^{\infty}\frac{(P_{k-1}(x) - P_{k+1}(x))^2}{2k+1}
\end{aligned}$$

and the lemma follows from inequality $|P_n(x)| \leq 1$ and (3.10).

Lemma 3. *For $-1 < x < t < 1$ we have*

$$|L_n(x,t)| \leq \left(\frac{K}{(t-x)^{1/2}} + \frac{M}{(1-x^2)^{1/4}}\right)\frac{1}{n^{3/2}} \tag{3.14}$$

and

$$|L_n(x,t)| \leq \left(\frac{K}{t-x} + \frac{M}{(1-x^2)^{1/4}(1-t^2)^{1/4}}\right)\frac{1}{n^2}. \tag{3.15}$$

Proof of Lemma 3. To prove inequality (3.14) of Lemma 3 in view of (3.5) and inequlities (3.7) and (3.8) it is sufficient to consider the function

$$W_n(x,t) = \sum_{k=n+1}^{\infty}\frac{(P_{k-1}(x) - P_{k+1}(x))(P_{k-1}(t) - P_{k+1}(t))}{2k+1}$$

and to show that

$$|W_n(x,t)| \leq K(t-x)^{-1/2}n^{-3/2}. \tag{3.16}$$

Writing $x = \cos\theta$, $t = \cos\psi$ we have, by (3.11) and (3.12),

$$\begin{aligned}
&(P_{k-1}(\cos\theta) - P_{k+1}(\cos\theta))(P_{k-1}(\cos\psi) - P_{k+1}(\cos\psi)) \\
&\qquad = 4\left(\frac{2\psi\,\sin\psi\; sin\theta}{\pi k}\right)^{1/2} J_1\left(\left(k+\frac{1}{2}\right)\psi\right)\sin\left(\left(k+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right) \\
&\qquad + O((\sin\theta)^{-1/2}k^{-2}).
\end{aligned}$$

Hence

$$\begin{aligned}
W_n(x,t) = 4\left(\frac{2\psi\,\sin\psi\,\sin\theta}{\pi}\right)^{1/2}&\sum_{k=n+1}^{\infty}\frac{1}{(2k+1)\sqrt{k}} \\
&J_1\left(\left(k+\frac{1}{2}\right)\psi\right)\sin\left(\left(k+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right) + O((1-x^2)^{-1/4}n^{-2}).
\end{aligned}$$

We shall next use the formula

$$J_1(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin(x\ \cos\nu)\cos\nu\, d\nu$$

(see [9, p. 48]). We have then

$$W_n(x,t) = 4\left(\frac{2\psi\ \sin\psi\ \sin\theta}{\pi}\right)^{1/2} \sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}}$$
$$\frac{2}{\pi}\left(\int_0^{\pi/2} \sin\left(\left(k+\frac{1}{2}\right)\psi\ \cos\nu\right)\cos\nu d\nu\right)\sin\left(\left(k+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right)$$

or

$$W_n(x,t) = \frac{8}{\pi}\left(\frac{2\psi\ \sin\psi\ \sin\theta}{\pi}\right)^{1/2} \int_0^{\pi/2} R_n(\psi\ \cos\nu,\,\theta)\cos\nu\, d\nu \tag{3.17}$$

where

$$\begin{aligned} R_n(\alpha,\theta) &= \sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}} \sin\left(\left(k+\frac{1}{2}\right)\alpha\right)\sin\left(\left(k+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}}\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}} \sin\left(\left(k+\frac{1}{2}\right)\alpha\right)\sin\left(\left(k+\frac{1}{2}\right)\theta\right. \\ &\quad - \frac{1}{\sqrt{2}}\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}} \sin\left(\left(k+\frac{1}{2}\right)\alpha\right)\cos\left(\left(k+\frac{1}{2}\right)\theta\right). \end{aligned}$$

Using well known trigonometric identities we find that

$$\begin{aligned} R_n(\alpha,\theta) = \frac{1}{2\sqrt{2}}&\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}} \cos\left(\left(k+\frac{1}{2}\right)(\alpha-\theta)\right) \\ - \frac{1}{2\sqrt{2}}&\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}} \cos\left(\left(k+\frac{1}{2}\right)(\alpha+\theta)\right) \\ - \frac{1}{2\sqrt{2}}&\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}} \sin\left(\left(k+\frac{1}{2}\right)(\alpha+\theta)\right) \\ - \frac{1}{2\sqrt{2}}&\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}} \sin\left(\left(k+\frac{1}{2}\right)(\alpha-\theta)\right). \end{aligned}$$

Since

$$\left|\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}} \cos\left(\left(k+\frac{1}{2}\right)x\right)\right| \leq \frac{1}{|\sin(x/2)|n^{3/2}}$$

and

$$\left|\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)\sqrt{k}} \sin\left(\left(k+\frac{1}{2}\right)x\right)\right| \leq \frac{1}{|\sin(x/2)|n^{3/2}}$$

and $0<\psi<\theta<\pi/2\,(x<t<1)$, it follows that, for $0\le\alpha\le\psi<\theta<\pi/2$,

$$\begin{aligned}|R_n(\alpha,\theta)| &\le \frac{1}{\sqrt{2}}\left(\frac{1}{\sin((\theta-\alpha)/2)}+\frac{1}{\sin((\theta+\alpha)/2)}\right)\frac{1}{n^{3/2}}\\ &\le \frac{\pi}{\sqrt{2}}\left(\frac{1}{\theta-\alpha}+\frac{1}{\theta+\alpha}\right)\frac{1}{n^{3/2}}\le\frac{2\theta\pi}{\sqrt{2}(\theta^2-\alpha^2)n^{3/2}}\le\frac{2\pi}{\sqrt{2}(\theta-\alpha)n^{3/2}}.\end{aligned}$$

Using the inequality and (3.17) we find that

$$|W_n(x,t)|\le\frac{8}{\pi}\left(\frac{2\psi\,\sin\psi\,\sin\theta}{\pi}\right)^{1/2}\frac{2\pi}{\sqrt{2}}\left(\int_0^{\pi/2}\frac{d\nu}{\theta-\psi\,\cos\nu}\right)\frac{1}{n^{3/2}}. \tag{3.18}$$

Since

$$\begin{aligned}\int_0^{\pi/2}\frac{d\nu}{\theta-\psi\,\cos\nu}&=2\int_0^1\frac{dz}{\theta-\psi+(\theta+\psi)z^2}\\ &\le 2\int_0^\infty\frac{dz}{\theta-\psi+(\theta+\psi)z^2}\le\frac{\pi}{(\theta^2-\psi^2)^{1/2}}\end{aligned}$$

it follows that

$$|W_n(x,t)|\le\frac{16\pi}{\sqrt{2}}\left(\frac{2\psi\,\sin\psi\,\sin\theta}{\pi}\right)^{1/2}\frac{1}{(\theta^2-\psi^2)^{1/2}n^{3/2}}.$$

Now

$$(\theta^2-\psi^2)/4=(\theta-\psi)/2\cdot(\theta+\psi)/2\ge\sin((\theta-\psi)/2)\sin((\theta+\psi)/2)\ge(\cos\psi-\cos\theta)/2.$$

So $|W_n(x,t)|\le 8\pi/(t-x)^{-1/2}n^{-3/2}$ and inequality (3.16) is proved. This completes the proof of inequality (3.14).

The proof of inequality (3.15) is similar. We have, by (3.12),

$$\begin{aligned}&(P_{k-1}(\cos\theta)-P_{k+1}(\cos\theta))(P_{k-1}(\cos\psi)-P_{k+1}(\cos\psi))\\ &\quad=\frac{8}{\pi k}(\sin\theta\,\sin\psi)^{1/2}\sin\left(\left(k+\frac{1}{2}\right)\theta-\frac{\pi}{4}\right)\sin\left(\left(k+\frac{1}{2}\right)\psi-\frac{\pi}{4}\right)+R_K(\psi,\theta)\end{aligned}$$

where $|R_K(\psi,\theta)|\le K(\sin\theta\,\sin\psi)^{-1/2}k^{-2}$. Hence

$$\begin{aligned}W_n(x,t)&=\frac{4}{\pi}(\sin\theta\,\sin\psi)^{1/2}\sum_{k=n+1}^{\infty}\frac{1}{k(2k+1)}\cos\left(\left(k+\frac{1}{2}\right)(\theta-\psi)\right)\\ &\quad-\frac{4}{\pi}(\sin\theta\,\sin\psi)^{1/2}\sum_{k=n+1}^{\infty}\frac{1}{k(2k+1)}\cos\left(\left(k+\frac{1}{2}\right)(\theta+\psi)\right)\\ &\quad+\sum_{k=n+1}^{\infty}\frac{1}{2k+1}R_K(\theta,\psi).\end{aligned}$$

Since, for $0<x<\pi$,

$$\left|\sum_{k=n+1}^{\infty}\frac{1}{k(2k+1)}\cos\left(\left(k+\frac{1}{2}\right)x\right)\right|\le\frac{1}{n^2\sin(x/2)}$$

we have

$$
\begin{aligned}
|W_n(x,t)| &\le \frac{4}{\pi}\left(\frac{1}{\sin((\theta-\psi)/2)}+\frac{1}{\sin((\theta+\psi)/2)}\right)\frac{1}{n^2}+\frac{K}{(\sin\theta\,\sin\psi)^{1/2}n^2}\\
&\le \frac{8}{\pi}\frac{1}{\sin((\theta-\psi)/2)\sin((\theta+\psi)/2)n^2}+\frac{K}{(\sin\theta\,\sin\psi)^{1/2}n^2}\\
&\le \frac{16}{\pi}\left(\frac{1}{\cos\psi-\cos\theta}\right)\frac{1}{n^2}+\frac{K}{(\sin\theta\,\sin\psi)^{1/2}n^2}
\end{aligned}
$$

or

$$
|W_n(x,t)| \le \frac{16}{\pi}\Big(\frac{1}{t-x}\Big)\frac{1}{n^2}+\frac{K}{((1-x^2)(1-t^2))^{1/4}n^2}
$$

and inequality (3.15) is proved. This completes the proof of Lemma 3.

4. *Proof of Theorem* 3B. We have

$$
\begin{aligned}
\int_x^1 g(t)B_n(x,t)\,dt &= \left(\int_x^{x+(1-x)/n}+\int_{x+(1-x)/n}^1\right)g(t)B_n(x,t)\,dt \\
&= I_n(g,x)+J_n(g,x).
\end{aligned}
\tag{4.1}
$$

We have first, since $g(x)=0$,

$$
\begin{aligned}
|I_n(g,x)| &\le \left(\int_x^{x+(1-x)/n}(g(t)-g(x))^2dt\right)^{1/2}\left(\int_x^{x+(1-x)/n}B_n^2(x,t)\,dt\right)^{1/2}\\
&\le V_x^{x+(1-x)/n}(g)n^{-1/2}(C/n)^{1/2}
\end{aligned}
$$

or

$$
|I_n(g,x)| \le Cn^{-1}V_x^{x+(1-x)/n}(g).
$$

Since $\sum_{k=1}^n \frac{1}{\sqrt{k}}V_x^{x+(1-x)/k}(g)\ge\sqrt{n}V_x^{x+(1-x)/n}(g)$ it follows that

$$
|I_n(g,x)| \le \frac{C}{n^{3/2}}\sum_{k=1}^n\frac{1}{\sqrt{k}}V_x^{x+(1-x)/k}(g).
\tag{4.2}
$$

Next we have

$$
J_n(g,x) = -\int_{x+(1-x)/n}^1 g(t)dL_n(x,t)
$$

where

$$
L_n(x,t)=\int_t^1 B_n(x,u)\,du=\int_t^1(\nu-t)K_n(x,\nu)\,d\nu.
$$

Integrating by parts we find that

$$
\begin{aligned}
J_n(g,x) &= (g(x+(1-x)/n)-g(x))L_n(x,x+(1-x)/n)\\
&\quad+\int_{x+(1-x)/n}^1 L_n(x,t)\,dg(t).
\end{aligned}
\tag{4.3}
$$

Using inequality (3.14) we find that

$$\begin{aligned}|J_n(g,x)| \le & V_x^{x+(1-x)/n}(g)\left(\frac{K\sqrt{n}}{(1-x^2)^{1/2}}+\frac{M}{(1-x^2)^{1/4}}\right)\frac{1}{n^{3/2}}\\ &+\frac{1}{n^{3/2}}\int_{x+(1-x)/n}^{1}\left(\frac{K}{(t-x)^{1/2}}+\frac{M}{(1-x^2)^{1/4}}\right)dV_x^t(g).\end{aligned}$$

Integrating the last integral by parts again, we find that

$$\begin{aligned}|J_n(g,x)| \le & \frac{1}{n^{3/2}}\left(\frac{K}{(1-x^2)^{1/2}}+\frac{M}{(1-x^2)^{1/4}}\right)V_x^1(g)\\ &+\frac{K}{2n^{3/2}}\int_{x+(1-x)/n}^{1}(t-x)^{-3/2}V_x^t(g)\,dt.\end{aligned}$$

Change of variable $t-x=(1-x)/\nu$ reduces the last integral to

$$\begin{aligned}\int_{x+(1-x)/n}^{1}(t-x)^{-3/2}V_x^t(g)\,dt &= (1-x)^{-1/2}\int_1^n \nu^{-1/2}V_x^{x+(1-x)/\nu}(g)\,d\nu\\ &\le (1-x^2)^{-1/2}\sum_{k=1}^{n}k^{-1/2}V_x^{x+(1-x)/k}(g).\end{aligned}$$

Consequently,

$$(1-x^2)^{1/2}|J_n(g,x)| \le Cn^{-3/2}\sum_{k=1}^{n}k^{-1/2}V_x^{x+(1-x)/k}(g) \tag{4.4}$$

and the proof of inequality (2.4B) of Theorem 3B follows from (4.1), (4.2) and (4.4).

The proof of second inequality (2.5B) of Theorem 3B is similar. The only difference is that when estimating $I_n(g,x)$ we conclude from

$$|I_n(g,x)| \le Cn^{-1}V_x^{x+(1-x)/n}(g)$$

and

$$\sum_{k=1}^{n}V_x^{x+(1-x)/k}(g) \ge nV_x^{x+(1-x)/n}(g)$$

$$|I_n(g,x)| \le Cn^{-2}\sum_{k=1}^{n}V_x^{x+(1-x)/k}(g). \tag{4.5}$$

Also, when estimating the integral

$$\int_{x+(1-x)/n}^{1}L_n(x,t)\,dg(t)$$

on the right-hand side of (4.3) we use inequality (3.15) for $L_n(x,t)$ instead of (3.14). We obtain thus the inequality

$$|J_n(g,x)| \le \frac{1}{n^2} V_x^{x+(1-x)/n}(g)\left(\frac{Kn}{1-x} + \frac{2M}{(1-x^2)^{1/2}}\right) + \frac{K}{n^2}\int_{x+(1-x)/n}^{1} \frac{1}{t-x} dV_x^t(g) + \frac{M}{(1-x^2)^{1/4}n^2}\int_x^1 (1-t^2)^{-1/4} dV_x^t(g).$$

Integrating the first integral by parts we find that

$$\int_{x+(1-x)/n}^{1} \frac{1}{t-x} dV_x^t(g) = \frac{1}{1-x} V_x^1(g) - \frac{n}{1-x} V_x^{x+(1-x)/n}(g) + \int_{x+(1-x)/n}^{1} (t-x)^{-2} V_x^t(g)\, dt.$$

So

$$\begin{aligned} |J_n(g,x)| \le & Kn^{-2}(1-x)^{-1}V_x^1(g) + 2M(1-x^2)^{-1/2}n^{-2}V_x^{x+(1-x)/n}(g) \\ & + Mn^{-2}(1-x^2)^{-1/4}\int_{x+(1-x)/n}^{1}(1-t^2)^{-1/4}dV_x^t(g) \\ & + Kn^{-2}\int_{x+(1-x)/n}^{1}(t-x)^{-2}V_x^t(g)\,dt \end{aligned}$$

and we find, as before, that

$$(1-x^2)|J_n(g,x)| \le Cn^{-2}\sum_{k=1}^{n} V_x^{x+(1-x)/k}(g) + Mn^{-2}\int_{x+(1-x)/n}^{1}(1-t^2)^{-1/4}dV_x^t(g). \tag{4.6}$$

Finally, using (4.1), (4.5) and (4.6) we obtain the second inequality of Theorem 3B.

REFERENCES

[1] E.W. Hobson, *On a general convergence theorem and the theory of the representation of functions by series of normal functions*, Proc. London Math. Soc. **6** (1908), 349–365.

[2] E.W. Hobson, *On the representation of a function by a series of Legendre's functions*, Proc. London Math. Soc. **7** (1909), 24–39.

[3] H. Burkhardt, *Zur Theorie der trigonometrischen Reihen und der Entwicklungen nach Kugel-funktionen*, Sitzungsberichte der Königl. Bayerischen Akademie der Wissenschfften, Math. Phys. Klasse, **39** (1909) 10 Abhandlung.

[4] R. Bojanić and M. Vuilleumier, *On the rate of convergence of Fourier-Legendre series of functions of bounded variation*, J. Approx. Theory **31** (1981), 67–79.

[5] Z. Divis, *An estimate for Fourier series of functions with derivatives of bounded variation*, Analysis, **6** (1986), 401–410.

[6] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc., New York, 1959.

[7] D. Jackson, *The Theory of Approximation*, Amer. Math. Soc., New York, 1930.

[8] G. Szegö, *Uber eine von Herrn S. Bernstein herrühende Abschätzung der Legendreschen Polynome*, Math. Anallen **108** (1933), 360–369.

[9] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge, 1944.

Department of Mathematics
Ohio State University
Columbus, Ohio 43210
USA

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