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INCLUSION RELATIONS BETWEEN SOME CLASES OF ALMOST HERMITE MANIFOLDS

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Abstract. A new method for obtaining conditions for almost Hemite manifolds is introduced. Almost Hermite manifolds contain Kählar manifolds, Tachibana, almost Kähler, quasi Kähler and Hermite manifolds. Inclusion relations betwen these manifolds are studied.

1. Introduction. An even dimensional differentiable manifold M^n is called an almost Hermite manifold if there are defined a tensor field f of type $(1,1)$ and metric tensor g, satisfying

$$f^2 + I = 0, \quad g(\bar{X}, \bar{Y}) = g(X, Y), \tag{1.1}$$

where $\bar{X} = fX$, and X, Y are elements of the Lie algebra $T(M^n)$ of vector fields on M^n.

Let ∇ be the Riemannian connexion. We can define a symmetric 2-covariant tensor field F by

$$F(M, N) = g(fM, N) \tag{1.2}$$

and we can consider its covariant derivate ∇F defined by

$$(\nabla_M F)(N, Q) \equiv \nabla F(M, N, Q) = g(\nabla_M fN, Q). \tag{1.3}$$

Then we have

a) $F(X,Y) = -F(Y,X)$ c) $(\nabla_X F)(\bar{Y}, \bar{Z}) = -(\nabla_X F)(Y, Z)$

b) $F(\bar{X}, \bar{Y}) = F(X, Y)$ d) $(\nabla_X F)(\bar{Y}, Z) = (\nabla_X F)(Y, \bar{Z})$.

Definition 1. An almost Hermite manifold is called a Kähler manifold if

$$(\nabla_X F)(Y, Z) = 0. \tag{1.4}$$

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Definition 2. An almost Hermite manifold is called an almost Tachibana manifold if

$$(1.5)\qquad (\nabla_X F)(Y,Z) - (\nabla_Y F)(Z,X) = 0.$$

Definition 3. An almost Hermite manifold is called an almost Kähler manifold if

$$(1.6)\qquad (\nabla_X F)(Y,Z) + (\nabla_Y F)(Z,X) + (\nabla_Z F)(X,Y) = 0.$$

Definition 4. An almost Hermite manifold is called a quasi Kähler manifold if

$$(1.7)\qquad (\nabla_X F)(Y,Z) + (\nabla_{\bar{X}} F)(\bar{Y},Z) = 0.$$

Definition 5. An almost Hermite manifold is called a Hermite manifold if

$$(1.8)\qquad (\nabla_X F)(Y,Z) - (\nabla_{\bar{X}} F)(\bar{Y},Z) = 0.$$

Conditions (1.4)–(1.8) will be called "Hermite conditions".

2. Hermite condition. Let us put $I \stackrel{\text{def}}{=} (\nabla_X F)(Y,Z)$, $\sigma \stackrel{\text{def}}{=} (\nabla_Y F)(Z,X)$. Then $\sigma^2 = (\nabla_Z F)(X,Y)$ and the following multiplication table holds

	I	σ	σ^2
I	I	σ	σ^2
σ	σ	σ^2	I
σ^2	σ^2	I	σ

Table 1

The system consisting of the set R_1 of all linear combinations of I, σ, σ^2 with multiplication as defined in Table 1 is an infinite comutative ring. If an element $a \in R_1$ is of the form $a = I + A\sigma + B\sigma^2$, then $a = 0$ is a Hermite condition. Now in R_1 we have $\sigma^3 - I = 0 \leftrightarrow (I-\sigma)\,(I+\sigma+\sigma^2) = 0$. Then four possibilities arise:

(1) $I - \sigma = 0$, $I + \sigma + \sigma^2 \neq 0$. The manifold is an almost Tachibana manifold.

(2) $I + \sigma + \sigma^2 = 0$, $I - \sigma \neq 0$. The manifold is an almost Kähler manifold.

(3) $I - \sigma = 0$, $I + \sigma + \sigma^2 = 0$. These equation yield $I = \sigma = \sigma^2 = 0$. The manifold is a Kähler manifold.

The intersection of the classes of almost Tachibana and almost Kähler manifolds is the class of Kähler manifolds.

(4) $I - \sigma \neq 0$, $I + \sigma + \sigma^2 \neq 0$. The manifold is neither almost Tachibena nor almost Kähler.

Since $I-\sigma=0$, $I+\sigma+\sigma^2=0$ are Hermite conditions, $(I-\sigma)^2=0$, $(I-\sigma)^3=0$, $\ldots$, $(I+\sigma+\sigma^2)^2=0$, $(I+\sigma+\sigma^2)^3=0$, $\ldots$, should also be Hermite conditions. We proceed to examine them. $(I-\sigma)^2=0 \leftrightarrow I-2\sigma+\sigma^2=0 \leftrightarrow \sigma^2-2I+\sigma=0_e$ since σ^2 admits a multiplicative inverse. From the last two equations we get $3(I-\sigma)=0$. Thus $(I-\sigma)^2=0$ is an almost Tachibana condition and so are all the other powers of $(I-\sigma)$. From the multiplicative table it can easily be established that $(I+\sigma+\sigma^2)^2=3(I+\sigma+\sigma^2)$. Thus $(I+\sigma+\sigma^2)^2=0$ is an almost Kählerian condition and so are the other powers of $I+\sigma+\sigma^2$.

THEOREM 2.1. *Put*

$$\alpha=(\nabla_{\bar{X}}F)(Y,Z),\ \ \beta=(\nabla_X F)(\bar{Y},Z),\ \ \gamma=(\nabla_{\bar{X}}F)(\bar{Y},Z)$$

Then I, α, β, γ admit the following multiplication table

	I	α	β	γ
I	I	α	β	β
α	α	$-I$	β	$-\beta$
β	β	γ	$-I$	$-\alpha$
γ	γ	$-\gamma$	$-\alpha$	I

Table 2

The system consisting of the set R_2 of all linear combinations of I, α, β, γ with multiplication as defined in Table 2 is an infinite comutative ring.

If $b=I+A\alpha+\beta B+C\gamma\in R_2$ then $b=0$ is a Hermite condition. We have the identity $I-\gamma^2=0\Leftrightarrow(I-\gamma)(I+\gamma)=0$. Again, four possibilities arise:

(1) $I+\gamma=0$, $I-\gamma\neq 0$. The manifold is quasi Kähler.

(2) $I-\gamma=0$, $I+\gamma\neq 0$. The manifold is Hermite.

(3) $I-\gamma=0$, $I+\gamma=0$. This gives $I=\gamma=0$. The manifold is Kähler manifold. Thus the intersection of the classes of quasi Kähler and Hermite manifolds is the class of Kähler manifolds.

(4) $I+\gamma\neq 0$, $I-\gamma\neq 0$. The manifold is neither quasi Kähler nor Hermite.

Also $(I+\gamma)^2=I^2+2\gamma+\gamma^2=2(I+\gamma)\cdot(I-\gamma)^2=2(I-\gamma)$. Thus the only "Hermite conditions" that can be obtained in R_2 are quasi Kähler and Hermite condition.

In R_2 we also have the identity

$$\alpha^2-\beta^2=0\Leftrightarrow(\alpha-\beta)(\alpha+\beta)=0.\ \text{But}\ \alpha-\beta=\alpha(I+\gamma),\ (\alpha+\beta)=\alpha(I-\gamma)$$

and $\alpha-\beta=0$ is a quasi Kähler condition. $\alpha+\beta=0$ is a Hermite condition.

$$(\alpha-\beta)^2=-2(I+\gamma),\ (\alpha+\beta)^2=-2(I-\gamma).\ \text{Then}\ (\alpha-\beta)^2=0,\ (\alpha+\beta)^2=0$$

give only a quasi Kähler and Hermite condition and so on.

Theorem 2.2. *$(I+\sigma+\sigma^2)(I+\gamma)=0$ is a quasi Kähler condition.*

Proof.

$$
\begin{aligned}
(I+\sigma+\sigma^2)(I+\gamma)=0 \Leftrightarrow &(\nabla_X F)(Y,Z)+(\nabla_Y F)(Z,X)+(\nabla_Z F)(X,Y)\\
&+(\nabla_{\bar X} F)(\bar Y,Z)+(\nabla_{\bar Y} F)(\bar Z,X)+(\nabla_{\bar Z} F)(\bar X,Y)=0. \qquad (2.1)
\end{aligned}
$$

For barring Y and Z in (2.1), we obtain

$$
\begin{aligned}
&-(\nabla_X F)(Y,Z)+(\nabla_{\bar Y} F)(\bar Z,X)+(\nabla_{\bar Z} F)(X,\bar Y)\\
&-(\nabla_{\bar X} F)(Y,\bar Z)+(\nabla_Y F)(Z,X)+(\nabla_Z F)(X,Y)=0. \qquad (2.2)
\end{aligned}
$$

Substracting (2.2) from (2.1), we obtain

$$(\nabla_X F)(Y,Z)+(\nabla_{\bar X} F)(\bar Y,Z)=0. \qquad (2.3)$$

Conversely, (2.3) satisfies (2.2) or (2.1). The class of almost Kähler manifolds is the subclass of the class of quasi Kähler manifolds.

Theorem 2.3. *$(I-\sigma)(I+\gamma)=0$ is a quasi Kähler condition.*

Proof.

$$
\begin{gathered}
(I-\sigma)(I+\gamma)=0 \Leftrightarrow\\
(\nabla_X F)(Y,Z)+(\nabla_{\bar X} F)(\bar Y,Z)-(\nabla_Y F)(Z,X)-(\nabla_{\bar Y} F)(Z,\bar X)=0. \qquad (2.4)
\end{gathered}
$$

For barring Y and Z, we obtain

$$-(\nabla_X F)(Y,Z)-(\nabla_{\bar X} F)(\bar Y,Z)-(\nabla_Y F)(Z,X)-(\nabla_{\bar Y} F)(Z,\bar X)=0. \qquad (2.5)$$

Subtracting (2.5) from (2.4), we obtain

$$(\nabla_X F)(Y,Z)+(\nabla_{\bar X} F)(\bar Y,Z)=0. \qquad (2.6)$$

Conversely, (2.6) satisfies (2.4) or (2.5). The class of almost Tachibana manifolds is the subclass of the class of quasi Kähler manifolds.

Let A and B be class of manifolds; then $A\Rightarrow B \Leftrightarrow A\subset B$.

Theorem 2.4. *Inclusions among almost Hermite manifolds are given by the following diagram*

$$
\begin{array}{ccc}
 & \textit{almost Kähler} & \\
 & \Uparrow \qquad \Downarrow & \\
\textit{Hermite} \Leftarrow & \textit{Kähler} \quad \textit{quasi Kähler} & \\
 & \Downarrow \qquad \Uparrow & \\
 & \textit{almost Tachibana} &
\end{array}
$$

3. The Nijenhuis tensor. We shall denote by N the Nijenhuis tensor defined by

$$N(X,Y,Z)=(\nabla_{\bar X} F)(Y,Z)+(\nabla_{\bar Y} F)(Z,X)+(\nabla_X F)(\bar Y,Z)+(\nabla_Y F)(\bar Z,X).$$

Let us put $M(X,Y,Z) = (\nabla_{\bar{X}}F)(Y,Z) + (\nabla_X F)(\bar{Y},Z)$; then we have

(a) $N(X,Y,Z) = \alpha+\beta+\alpha\sigma+\beta\sigma = (\alpha+\beta)(I+\sigma)$ (b) $M(X,Y,Z) = \alpha+\beta$.

From the last equation it easily follows that $N(X,Y,Z) = M(X,Y,Z)+M(Y,Z,X)$.

THEOREM 3.1. *On a quasi Kähler manifold $\alpha = \beta$. Hence $N(X,Y,Z) = 2\alpha(I+\sigma)$, $M(X,Y,Z) = 2\alpha = 2\beta$. The necessary and sufficient condition for $N = 0$ on a quasi Kähler manifold is that it reduces to a Kähler manifold.*

The results obtained so far in this paper are known but they are obtained here in a much simpler way, using a new method. We proceed to give some new theorems. From (a) and (b) it follows that

THEOREM 3.2 $2M(X,Y,Z) = 2(\alpha+\beta) = (\alpha+\beta)(I+\sigma)(I-\sigma+\sigma^2) = = (I-\sigma+\sigma^2)\cdot N(X,Y,Z) = N(X,Y,Z) - N(Y,Z,X) + N(Z,X,Y)$.

THEOREM 3.3. $M(\bar{X},Y,Z) = -I+\gamma = M(X,\bar{Y},Z) = M(X,Y,\bar{Z})$. $M(\bar{X},\bar{Y},Z) = M(\bar{X},Y,\bar{Z}) = M(X,\bar{Y},\bar{Z}) = -M(X,Y,Z)$.

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