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CONDITIONS ON THE CONHARMONIC CURVATURE TENSOR OF KAEHLER HYPERSURFACES IN COMPLEX SPACE FORMS

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Abstract. Among all Kaehler hypersurfaces of complex space forms, we characterize the complex linear hyperplanes, the complex hypercylindes in the complex Euclidean spaces and the complex hyperquadrics in the complex projective spaces, in terms of some intrinsic curvature conditions which all involve the conharmonic curvature tensor of these hypersurfaces.

1. Introduction. Let M^n a Kaehler hypersurface of complex dimension n in a complete simply connected Kaehler manifold of constant holomorphic sectional curvature $\tilde{M}^{n+1}(c)$ (i.e. in a *complex space form* $\tilde{M}^{n+1}(c)$). It is known that a complex space form $\tilde{M}^{n+1}(c)$ is holomorphically isometric with $\mathbf{C}P^{n+1}(c)$, $\mathbf{C}^{n+1}$ or $\mathbf{D}^{n+1}(c)$, according to c being positive, zero or negative $\cdot\mathbf{C}P^{n+1}(c)$ is the complex projective space with the Study-Fubini metric of holomorphic sectional curvature c, $\mathbf{C}^{n+1}$ the complex Euclidean space and $\mathbf{D}^{n+1}(c)$ the unit ball all in $\mathbf{C}^{n+1}$ with the Bergman metric of holomorphic sectional curvature c. By setting one coordinate equal to zero, $\mathbf{C}P^n(c)$, $\mathbf{C}^n$ and $\mathbf{D}^n(c)$ occur naturally as totally geodesic complex hypersurfaces in $\mathbf{C}P^{n+1}(c)$, $\mathbf{C}^{n+1}$ and $\mathbf{D}^{n+1}(c)$, respectively; these hypersurfaces are also called the *complex hyperplanes* of $\mathbf{C}P^{n+1}(c)$, $\mathbf{C}^{n+1}$ and $\mathbf{D}^{n+1}(c)$, respectively. The *complex hypersphere* Q^n in $\mathbf{C}P^{n+1}(c)$ is the complex hypersurface with equation $z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0$, where $z_0, z_1, \dots, z_{n+1}$ are homogenous coordinates; Q^n is a locally symmetric Einstein space. A *complex hypercylinder* $\mathcal{B}^n$ in $\mathbf{C}^{n+1}$ is the product of an $(n-1)$-dimensional complex hyperplane $\mathbf{C}^{n-1}$ in $\mathbf{C}^n \subset \mathbf{C}^{n+1}$ with a complex curve in a 2-dimensional complex plane orthogonal to $\mathbf{C}^{n-1}$ in $\mathbf{C}^{n+1}$. According to a result of K. Abe [1], a complete complex hypersurface in a complex Euclidean space for which rank $(A) \leq 2$ is a complex hypercylinder, where A denotes the second fundamental tensor of the hypersurface.

It is a classical theme in differential geometry to study relations between intrinsic and extrinsic properties of submanifolds. Therefore the purpose of this paper is to classify those hypersurfaces M^n which satisfy one of the following conditions: $K \circ Q = 0$, $Q \circ K = 0$, $K \circ R = 0$, $R \circ K = 0$, $K \circ C = 0$, $C \circ K = 0$, $P \circ K = 0$,

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$K \circ P = 0$, $B \circ K = 0$, $K \circ B = 0$, $K \circ G = 0$, $G \circ K = 0$, where R, C, P, B, Q, G and K are respecitvely the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Weyl projective curvature tensor, the Bochner curvature tensor, the Ricci endomorphism, the Einstein tensor and conharmonic curvature tensor of M^n, and where the first tensor acts on the second as a derivation.

For hypersurfaces in Euclidean spaces, K. Nomitzu initiated the study of such curvature conditions in his paper [12]. For any pair X and Y of vector fields in Riemannian manifold, $R(X,Y)$ is an skew-symmetric endomorphism of the tangent space at each point. The mapping of the algebra of tensor fields into itself given by

$$T \to \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T$$

is the unique derivation which extends $R(X,Y)$ and which commutes with all contractions. Thus it is natural to write $R(X,Y) \circ T$ for the image of an arbitrary tensor field T under this mapping. In particular, Nomizu considered the $(1,3)$ tensor field $R(X,Y) \circ R$. It acts on a pair of vector fields U and V as follows

$$(R(X,Y) \circ R)(U,V) = [R(X,Y),\, R(U,V)] - R(R(X,X)U,V) - R(U,R(X,Y)V).$$

In [12], K. Nomizu studied certain hypersurfaces of Euclidean spaces which are semi-symmetric, i.e. for which $R \circ R = 0$ holds, and proved they are also locally symmetric; he then raised the question whether or not, in general, semi-symmetry implies local symmetry. In this respect, H. Takagi was the first to find a counterexample, and afer that S. Tanno, K. Sekigawa, H. Tagaki, P. Ryan, Z.I. Szabo and others proceeded to investigate the implication of the condition $R(X,Y) \circ R = 0$ for some special manifolds or submanifolds. The authors B. Smyth, T. Takahashi, K. Nomitzu, P. Ryan, H. Tagaki, Y. Watanabe and others have been studying the curvature conditions on complex manifolds. P. Verheyen, J. Deprez, D.E. Blair, F. Dillen and two of the present authors obtained classification theorems for some special manifolds, in terms of the above type of conditions involving various curvature tensors (Ricci tensor Q, Bochner curvature tensor B, Weyl conformal curvature tensor C, projective curvature tensor P, concircular curvature tensor Z etc). In particular, in [5] and [7], they obtained some non-trivial hypersurfaces of revolution in Euclidean spaces satisfying such curvature conditions, amongst which we quote the 3- and 4-dimensional hypercatenoids. The articles of the above mentioned authors about these curvature conditions are cited in the references of [8].

The purpose of this article is to investigate the above type of curvature conditions, which involve the conharmonic curvature tensor, for complex hypersurfaces in complex space forms. More precisely, we want to find out whether such curvature conditions give some new characterizations of the well-known classes of such Kaehler hypersurfaces or whether eventually some new classes of such hypersurfaces appear in this way. We will recall some basic notions from the paper [14] of B. Smyth, who initiated the study of complex hypersurfaces.

The *conharmonic curvature tensor* K of a Riemannian manifold M was introduced by Y. Ishii [9]. K is invariant under the action of the conformal transformations of M which preserve, in a certain sense, real harmonic functions on M, and which therefore are called conharmonic transformations. These transformations form a subgroup of the conformal transformation group. A Riemannian manifold M is related to a Euclidean space of the same dimension by a conharmonic transformation, and then is said to be conharmonically Euclidean, if and only if $K = 0$, i.e. when K vanishes identically. In this respect, one has the following result.

THEOREM A. ([7]) *A Riemannian manifold M of dimension ≥ 4 is conharmonically Euclidean if and only if M is (locally) conformally Euclidean* (i.e. $C = 0$) *and M has zero scalar curvature.*

Conditions of the form $E \circ F = 0$ where a curvature tensor E on a complex hypersurface M^n of a complex space form $\tilde{M}^{n+1}(c)$ acts as a derivation on a curvature tensor F on M^n, were studied amongst others by P.J. Ryan [13], one of the authors and J. Deprez and P. Verheyen [6], and two of the authors and J. Deprez and F. Dillen [4]. Concerning curvature conditions on complex hypersurfaces, we recall the following results.

THEOREM B. ([16], [6]) *For a Kaehler hypersurface M^n of dimension $n \geq 2$ in a complex space form $\tilde{M}^{n+1}(c)$, the following statements are equivalent:*

(1) *M^n is flat* (i.e. $R = 0$)*;*

(2) *M^n is (locally) conformally Euclidean;*

(3) *M^n is Ricci flat (i.e. $Q = 0$);*

(4) *$c = 0$ and M^n is totally geodesic, i.e. M^n is a complex linear hyperplane in the complex number space $\mathbf{C}^{n+1}$.*

From Theorem A and Theorem B we have the following.

COROLLARY C. *The only complex hypersurfaces M^n of compex space forms $\tilde{M}^{n+1}(c)$ with $n \geq 2$ which are conharmonically Euclidean are the complex linear hyperplanes in the complex number space $\mathbf{C}^{n+1}$.*

Finally, we also recall the following results.

THEOREM D. ([11], [6]) *A Kaehler hypersurface M^n of dimension $n \geq 2$ in a complex space form $\tilde{M}^{n+1}(c)$ is Bochner flat* (i.e. $B = 0$)*, if and only if it is totally geodesic (in particular, M^n then is itself a complex space form of constant holomorphic sectional curvature c).*

THEOREM E. ([3]) *A Kaehler hypersurface M^n of dimension $n \geq 2$ in a complex space form $\tilde{M}^{n+1}(c)$ is Einsteinian (*i.e. *Q is proportional to the identity, or still $G = 0$), if and only if M^n is totally geodesic in $\tilde{M}^{n+1}(c)$ or M^n is locally a complex hypersphere (or hyperquadric) Q^n in the complex projective space $\mathbf{C}P^{n+1}(c)$.*

THEOREM F. ([6]) *A complete Kaehler hypersurface M^n of dimension $n \geq 2$ in a complex space form $\tilde{M}^{n+1}(c)$ satisfies the curvature condition $R \circ C = 0$ if and only if M^n is a complete totally geodesic hypersurface in $\tilde{M}^{n+1}(c)$ or M^n is a complex hypersphere Q^n in $\mathbf{C}P^{n+1}(c)$ or M^n is a complex hypercylinder $\mathcal{B}^n$ in $\mathbf{C}^{n+1}$.*

In the present paper, we will prove the following results.

THEOREM 1. *For a complete Kaehler hypersurface M^n with $n \geq 2$ in a complex space form $\tilde{M}^{n+1}(c)$, $R \circ K = 0$ if and only if M^n is a complete totally geodesic hypersurface in $\tilde{M}^{n+1}(c)$ or M^n is a complex hypersphere Q^n in $\mathbf{C}P^{n+1}(c)$ or M^n is a complex hypercylinder $\mathcal{B}^n$ in $\mathbf{C}^{n+1}$.*

THEOREM 2. *For a Kaehler hypersurface M^n with $n \geq 2$ in a complex space form $\tilde{M}^{n+1}(c)$, the following assertions are equivalent:* (1) $G \circ K = 0$; (2) $K \circ G = 0$; (3) $K \circ Q = 0$; (4) *M^n is Einsteinian.*

THEOREM 3. *For a Kaehler hypersurface M^n with $n \geq 2$ in a complex space form $\tilde{M}^{n+1}(c)$, the following assertions are equivalent:* (1) $B \circ K = 0$; (2) $K \circ B = 0$; (3) *M^n is Bochner flat.*

THEOREM 4. *For a Kaehler hypersurface M^n with $n \geq 2$ in a complex space form $\tilde{M}^{n+1}(c)$, the following assertions are equivalent:* (1) $K \circ R = 0$; (2) $K \circ C = 0$; (3) $C \circ K = 0$; (4) $K \circ P = 0$; (5) $P \circ K = 0$; (6) $Q \circ K = 0$; (7) $K \circ K = 0$; (8) *M^n is flat.*

For more results of this type, involving amongst others also the holomorphic concircular curvature tensor and the holomorphic projective curvature tensor of the complex hypersurfaces, we refer to [8]. Hypersurfaces of Euclidean spaces satisfying similar curvature conditions with the tensor K are studied in [2].

2. Basic formulas. Let M^n be a connected complex manifold of complex dimension n holomorphically immersed in Kaehler manifold $\tilde{M}^{n+1}$ of complex dimension $n+1$, i.e. a *complex hypersurface* in $\tilde{M}^{n+1}$. The complex structure J and the Kaehler metric g of $\tilde{M}^{n+1}$ induce, respectively, a complex structure and a Kaehler metric on M^n [14]. We denote these induced objects by the same letters. For each $x_0 \in M^n$, we choose a smooth field of unit normals ξ defined in a neighborhood $\mathcal{U}$ of x_0. Denoting by $\tilde{\nabla}$ the Kaehlerian connection on $\tilde{M}^{n+1}$, we have, for vector fields tangent to M^n in $\mathcal{U}$,

$$\tilde{\nabla}_x Y = \nabla_x Y + g(AX,Y)\xi + g(JAX,Y)J\xi \ \text{ and } \ \tilde{\nabla}_x \xi = -AX + s(X)J\xi,$$

where A is a symmetric tensor field of type (1, 1) on $\mathcal{U}$, called the *second fundamental form*, and ∇ is induced Kaehler connection on M^n. It is easy to show that $AJ = -JA$. One can look at [14] to find the proofs of this.

The *Riemann-Christoffel curvature tensor* R of M^n is expressed by the Gauss equation

$$R(X,Y) = \tilde{R}(X,Y) + AX \wedge AY + JAX \wedge JAY \tag{1}$$

where $\tilde{R}$ is the curvature tensor of $\tilde{M}^{n+1}(c)$, $X, Y \in T_p\tilde{M}$, $p \in \tilde{M}$ and where the symbol $\wedge$ is used in the following sense: $X \wedge Y$ is the skew-symmetric linear transformation defined by:

$$(X \wedge Y)Z = g(Z,Y)X - g(Z,X)Y.$$

Although A depends on our choice ξ, it is not difficult to show that A^2 is independent of this choice and so is defined on all of M^n as a tensor field of type $(1,1)$.

A two-dimensional subspace II of the (real) tangent space is called a *holomorphic plane* if there is a unit vector X such that X and JX span II. A Kaehler manifold $\tilde{M}$ is said to have constant *holomorphic sectional curvature* c if the number

$$K(\Pi) = g(\tilde{R}(X, JX)JX, X)$$

is equal to c for every holomorphic plane Π at every point of M^n. It is well known that this is true if and only if

$$\tilde{R}(X, X) = c(X \wedge Y + JX \wedge JY + 2g(X, JY)J)/4 \tag{2}$$

holds for all tangent vectors X and Y (see [10]).

For any $x \in M^n$, the tangent space to M^n at x is a 2n-dimensional real vector space with inner product g and complex structure J. The facts that $g(JX, JY)$ and $AJ = -JA$ lead to the existence of an orthonormal basis of eigenvectors of A which takes the form $\{e_i, Je_i\}_{i=1}^n$. For details, see Lemma 1 of [14]. We will use the notation $Je_i = e_{1^*}, \ldots, Je_n = e_{n^*}$. We also may choose the ordering so that, for $1 \leq i \leq n$, $A_i = \lambda_i e_{i^*}$ and $Ae_{i^*} = -\lambda_i e_{i^*}$, $\lambda_i \geq 0$. It can be shown (like in [15]) that the number of distinct eigenvalues is constant, that the multiplicities of the eigenvalues are constant and that the eigenvalue functions are differentiable.

The *Ricci endomorphism* Q of M^n is given by

$$Q = ((n+1)cI)/2 - 2A^2, \tag{3}$$

where I denotes the identity operator, and the *scalar curvature* τ is given by

$$\tau = n(n+1)c - 2Tr(A^2) = n(n+1) - 4\sum_{i=1}^{n} \lambda_i^2. \tag{4}$$

In the sequel, we will mostly identify the *Ricci tensor* S with the Ricci endomorphism Q, having $S(X, Y) = g(QX, Y) = g(X, QY)$.

The *Einstein tensor* G of M^n is defined by

$$G(X, Y) = S(X, Y) - \tau g(X, Y)/2n. \tag{5}$$

We also note, that we will mostly use the same notation G for the Einstein tensor G and for the corresponding *Einstein endomorphism* E, having $G(X, Y) = g(EX, Y) = g(X, EY)$.

The *Weyl conformal curvature tensor* C is given by

$$C(X, Y) = R(X, Y) - \frac{1}{2(n-1)}(QX \wedge Y + X \wedge QY) + \frac{\tau}{2(n-1)(2n-1)}(X \wedge Y), \tag{6}$$

the *Weyl projective curvature tensor* P by

$$P(X, Y) = R(X, Y) - (X \wedge Y)Q/(2n-1), \tag{7}$$

the *conharmonic curvature tensor* K by

$$K(X,Y) = R(X,Y) - (QX \wedge Y + X \wedge QY)/2(n-1), \tag{8}$$

and the *Bochner curvature tensor* B by

$$\begin{aligned} B(X,Y) &= R(X,Y) \\ &- \frac{1}{2(n+2)}(QX \wedge Y + X \wedge QY + QJX \wedge JY + JX \wedge QJY - 2g(QJX,Y)J \\ &- 2g(JX,Y)QJ) + \frac{\tau}{4(n+1)(n+2)}(X \wedge Y + JX \wedge JY + 2g(X,JY)J), \end{aligned} \tag{9}$$

for all vectors X and Y tangent to M^n at the same point. It is easy to check that these curvature tensors satisfy the following relations:

$$\begin{aligned} &R(X,Y) = R(JX,JY),\ R(X,Y)J = JR(X,Y),\ QJ = JQ,\ GJ = JG, \\ &C(JX,JY) = -JC(X,Y)J,\ P(JX,JY) = -JP(X,Y)J, \\ &K(JX,JY) = -JK(X,Y)J,\ B(JX,JY) = B(X,Y),\ B(X,Y)J = JB(X,Y). \end{aligned}$$

Then, with respect to an orthonormal basis $\{e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*}\}$ of the tangent space $T_p(M^n)$, we have the following formulas:

$$\begin{aligned} &Qe_i = \mu_i e_i,\ Qe_{i^*} = \mu_i e_{i^*},\ \text{ where } \mu_i = (n+1)c/2 - 2\lambda_i; \\ &Ge_i = g_i e_i,\ Ge_{i^*} = g_i e_{i^*},\ \text{ where } g_i = \mu_i - \tau/2n, \\ &R(e_i,e_j) = \nu_{ij}(e_i \wedge e_j + e_{i^*} \wedge e_{j^*})\ (i \neq j), \\ &R(e_i,e_{j^*}) = \bar\nu_{ij}(e_i \wedge e_{j^*} - e_{i^*} \wedge e_j) - c\delta_{ij}J/2, \end{aligned}$$

where $\nu_{ij} = c/2 + \lambda_i\lambda_j$ and $\bar\nu_{ij} = c/2 - \lambda_i\lambda_j$;

$$\begin{aligned} &C(e_i,e_j) = (\nu_{ij} + \alpha_{ij})e_i \wedge e_j + \nu_{ij} e_{i^*} \wedge e_{j^*}\ \ (i \neq j), \\ &C(e_i,e_{j^*}) = (\bar\nu_{ij} + \alpha_{ij})e_i \wedge e_{j^*} - \bar\nu_{ij} e_{i^*} \wedge e_j - \frac{c}{2}\delta_{ij}J, \end{aligned}$$

where $\alpha_{ij} = \dfrac{\tau}{2(n-1)(2n-1)} - \dfrac{1}{2(n-1)}(\mu_i + \mu_j)$;

$$\begin{aligned} &P(e_i,e_j)e_k = (\nu_{ij} - \mu_k/(2n-1))(\delta_{kj}e_i - \delta_{ki}e_j), \\ &P(e_i,e_j)e_{k^*} = \nu_{ij}(\delta_{kj}e_{i^*} - \delta_{ki}e_{j^*}), \\ &P(e_i,e_{j^*})e_k = -(\bar\nu_{ij} - \mu_k/(2n-1)\delta_{ik}e_{j^*} - \bar\nu_{ij}\delta_{jk}e_{i^*} - c\delta_{ij}e_{k^*}/2, \\ &P(e_i,e_{j^*})e_{k^*} = (\bar\nu_{ij} - \mu_k/(2n-1)\delta_{jk}e_i + \bar\nu_{ij}\delta_{ik}e_j + c\delta_{ij}e_k/2, \\ &K(e_i,e_j) = (\nu_{ij} + k_{ij})e_i \wedge e_j + \nu_{ij}e_{i^*} \wedge e_{j^*},\ (i \neq j), \\ &K(e_i,e_{j^*}) = (\bar\nu_{ij} + k_{ij})e_i \wedge e_{j^*} - \bar\nu_{ij}e_{i^*} \wedge e_j - c\delta_{ij}J/2, \end{aligned}$$

where $k_{ij} = -(\mu_i + \mu_j)/2(n-1)$;

$$\begin{aligned} &B(e_i,e_j) = \beta_{ij}(e_i \wedge e_j + e_{i^*} \wedge e_{j^*}),\ (i \neq j), \\ &B(e_i,e_{j^*}) = \bar\beta_{ij}(e_i \wedge e_{j^*} - e_{i^*} \wedge e_j) + \delta_{ij}(k_i + Q/(n+2))J, \end{aligned}$$

where

$$\beta_{ij} = \nu_{ij} - \frac{1}{2(n+2)}(\mu_i + \mu_j) + \frac{\tau}{4(n+1)(n+2)},$$
$$\bar{\beta}_{ij} = \bar{\nu}_{ij} - \frac{1}{2(n+2)}(\mu_i + \mu_j) + \frac{\tau}{4(n+1)(n+2)},$$
$$k_i = -\frac{c}{2} - \frac{\tau}{2(n+1)(n+2)} + \frac{1}{n+2}\mu_i$$

for all $i, j \in \{1, \ldots, n\}$.

From [6] we quote the following result.

Lemma A. *Let M^n be a complex hypersurface of $\tilde{M}^{n+1}(c)$. If for all indices $i, j \in \{1, \ldots, n\}$ we have*

$$(*) \qquad \lambda_i \lambda_j (\lambda_i^2 - \lambda_j^2) = 0,$$

then there exists a number k in $\{0, \ldots, n\}$ such that

$$\lambda_1 = \cdots = \lambda_k = \lambda \in R_0^+, \ \lambda_{k+1} = \cdots = \lambda_n = 0.$$

The case $k = 0$, i.e. $\lambda_1 = \cdots = 0$, means that M^n is totally geodesic; the case $k = n$ implies $c > 0$ and $\lambda^2 = c/4$ and means that M^n is locally a hypersphere in $\mathbf{C}P^{n+1}(c)$, while the case $c = 0$ and $k = 1$ means that M^n is a hypercylinder in $\mathbf{C}^{n+1}$.

Concerning the notations $R \circ K$, $Q \circ K$, $P \circ K$, ... in the introduction we recall that in these cases R, Q, P, ... act as derivations on the algebra of tensor fields on M^n which commute with contractions. For instance:

$$\begin{aligned}(R(X,Y) \circ K)(U,V)W &= R(X,Y)(K(U,V)W) - K(R(X,Y)U,V)W \\ &\quad - K(U,R(X,Y)V)W - K(U,V)(R(X,Y)W), \\ (Q \circ K)(X,Y)U &= Q(K(X,Y)U) - K(QX,Y)U - K(X,QY)U - K(X,Y)(QU), \\ (K(X,Y) \circ Q)U &= K(X,Y)(QU) - Q(K(X,Y)U),\end{aligned}$$

for all X, Y, U, V, W tangent to M^n.

We will also mention the following lemmas which can be verified by a straightforward computation.

Lemma B. *If T is a fourth order curvature tensor, then $T \circ M = 0$ where $M(X,Y) = X \wedge Y$.*

Lemma C. *If T is a fourth order curvature tensor, then $T \circ G = T \circ Q$.*

Lemma D. *If T is a fourth order curvature tensor, then the following conditions are equivalent: $T \circ P = 0$, $T \circ R = 0$.*

These results enable us to reduce the proofs of theorems in the next section. Since each of the tensors R, P, C and K can play the role of the tensor T in Lemma B, we obtain the following identities: $R \circ K = R \circ C$, $C \circ K = C \circ C$ and $K \circ K = K \circ C$. It also follows from Lemma D, that the condition $K \circ P = 0$ is equivalent to condition $K \circ R = 0$.

3. Proofs of Theorems. *Proof of Theorem* 1. Using the Lemma B we conclude that $R \circ K = R \circ C$ and from Theorem 1 [6] we immediately get the proof of Theorem 1.

Proof of Theorem 2. If M^n is Einstain hypersurface, then obviously conditions (1) and (2) are satisfied. Using Lemma C we get that the condition (3) is also satisfied.

Assume $G \circ K = 0$ and let i, j be distinct indices. Then we have

$$(10) \qquad (\nu_{ij} + k_{ij})g_i = 0, \qquad (11)\ \nu_{ij} g_i = 0, \qquad (12)\ \bar{\nu}_{ij} g_i = 0,$$

since $(G \circ K)(e_i, e_j)e_i = (G \circ K)(e_i, e_j)e_{i^*} = (G \circ K)(e_i, e_{j^*})e_{i^*} = 0$. Contrary to the statement, suppose that $G \neq 0$. Adding relationns (11) and (12) we get

$$(13) \qquad cg_i = 0,$$

whence $c = 0$ since $G \neq 0$. Hence, relation (11) becomes

$$(14) \qquad \lambda_i \lambda_j g_i = 0\, (i \neq j).$$

Substracting relations (10) and (11) we get

$$(15) \qquad k_{ij} g_i = 0,\ (i \neq j).$$

Interchanging indices i, j in (14) and subtracting, we get relation $(*)$ from Lemma A. Since $G \neq 0$, we get $0 < k < n$, where k is the number existing in Lemma A. Since then $g_1 = 2(n-k)\lambda^2/n \neq 0$ and $k_{1n} = -(\mu_1 + \mu_n)/2(n-1) = \lambda^2/(n-1) \neq 0$, relation (15) with indices $i = 1$, $j = n$ gives a contradiction. Now, assume that $K \circ Q = 0$, $G \neq 0$ and let i, j be distinct indices. Then we have

$$(16)\ \nu_{ij}(\mu_i - \mu_j) = 0, \qquad (17)\ \bar{\nu}_{ij}(\mu_i - \mu_j) = 0, \qquad (18)\ (\nu_{ij} + k_{ij})(\mu_i - \mu_j) = 0,$$

since $(K(e_i, e_j) \circ Q)e_{i^*} = (K(e_i, e_{j^*}) \circ Q)e_{i^*} = (K(e_i, e_j) \circ Q)e_i = 0$. Adding relations (16) and (17) we get

$$(19) \qquad c(\mu_i - \mu_j) = 0\, (i \neq j),$$

which since $G \neq 0$ gives $c = 0$. Hence, relation (16) becomes relation $(*)$. Since $G \neq 0$ we also get $0 < k < n$. But then, relation (18) with $\lambda_1 = \lambda > 0$ and $\lambda_n = 0$ for $i = 1$ and $j = n$, gives a contradiction.

As we know from Lemma C, $K \circ G = K \circ Q$ and this completes the proof of Theorem 2.

Proof of Theorem 3. Using Theorem D it is sufficient to prove that M^n is a totally geodesic hypersurface. The proof of $B \circ K = 0 \Leftrightarrow M^n$ is a totally geodesic hypersurface, is identical to the corresponding proof for $B \circ C = 0$ in [6].

When $K \circ B = 0$ it follows from $(K(e_i, e_j) \circ B)(e_i, e_{j^*})e_i = 0$, $(K(e_j, e_i) \circ B)(e_j, e_{i^*})e_i = 0$, $(K(e_i, e_{j^*}) \circ B)(e_i, e_j)e_i = 0$, and $(K(e_j, e_{i^*}) \circ B)(e_j, e_i)e_i = 0$, for distinct i and j that

$$\nu_{ij}[2(\bar{\beta}_{ii} - \bar{\beta}_{ij}) + k_i - k_j] + k_{ij}(k_j + \mu_i/(n+2) - \bar{\beta}_{ij}) = 0 \tag{20}$$

$$\nu_{ij}[2(\bar{\beta}_{ii} - \bar{\beta}_{ij}) + k_j - k_i] + k_{ij}(2\bar{\beta}_{ii} - \bar{\beta}_{ij} - k_i - \mu_i/(n+2)) = 0 \tag{21}$$

$$\bar{\nu}_{ij}[2(\bar{\beta}_{ii} - \beta_i) + k_j - k_i] + k_{ij}(k_j + \mu_i/(n+2) - \beta_{ij}) = 0 \tag{22}$$

$$\bar{\nu}_{ij}[2(\bar{\beta}_{ii} - \beta_{ij}) + k_j - k_i] + k_{ij}(2\bar{\beta}_{ii} - \beta_{ij} - k_i - \mu_i/(n+2)) = 0. \tag{23}$$

Addition of (20) and (21) implies

$$(2\nu_{ij} + k_{ij})(\lambda_j - \lambda_i)(n\lambda_i - 2\lambda_j) = 0. \tag{24}$$

Interchanging i and j in (24) and adding, we get

$$(2\nu_{ij} + k_{ij})(\lambda_i - \lambda_j)^2 = 0. \tag{25}$$

Subtracting (20) and (21) we get

$$k_{ij}(2\mu_i/(n+2) + k_i + k_j - 2\bar{\beta}_{ii}) = 0. \tag{26}$$

Interchanging i and j in (26) and subtracting we get

$$k_{ij}(\lambda_i - \lambda_j)(\lambda_i + \lambda_j)(n-2) = 0. \tag{27}$$

Addition of (22) and (23) gives

$$(2\bar{\nu}_{ij} + k_{ij})(\lambda_i + \lambda_j)(n\lambda_i + 2\lambda_j) = 0. \tag{28}$$

Inerchanging i and j in (28) and adding, we get

$$(2\bar{\nu}_{ij} + k_{ij})(\lambda_i + \lambda_j)^2 = 0. \tag{29}$$

Multiplying (25) by $(\lambda_i + \lambda_j)^2$ and (29) by $(\lambda_i - \lambda_j)^2$ and subtracting obtained relations we get the relation $(*)$.

Let us consider the case $k = n$. From the condition $\lambda^2 = c/4$ we get that $\bar{\nu}_{ij} = 0$ and hence from (29) it follows $k_{ij} = 0$. Then it follows that $c = 0$, i.e. $\lambda = 0$ which is a contradiction. If $k = 0$ then M^n is a totally geodesic hypersurface. Finally, we consider the case $0 < k < n$. If we put $i = \alpha$, $j = x$ in (27), we obtain

$$k_{\alpha x}(n-2)\lambda^2 = 0. \tag{30}$$

Now, we have two possibilities: $k_{\alpha x} = 0$ or $n = 2$. If $k_{\alpha x} = 0$ it follows that $\lambda^2 = (n+1)c/2$. In this case, relation (25) gives the contradiction $c = 0$, i.e. $\lambda = 0$. If $n = 2$, i.e. $\lambda_1 = \lambda \neq 0$ and $\lambda_2 = 0$, we obtain that $\lambda^2 = c$ from the relation (25). Using these results, from relation (20) we get $c = 0$, i.e. $\lambda = 0$, which is the desired contradiction.

The implication (3) $\Rightarrow$ (2) of course is trivial.

Proof of Theorem 4. Each flat hypersurface M^n obviously satisfies each of the conditions (1) – (7).

Next, assume $K \circ R = 0$. Let i, j be distinct indices; then we have

$$\lambda_i \lambda_j (2\bar{\nu}_{ii} + k_{ii}) = 0, \tag{31}$$

$$(c/2 + \bar{\nu}_{ij})(\nu_{ij} + k_{ij}) + (\bar{\nu}_{ij} - c/2 - 2\bar{\nu}_{ij})\nu_{ij} = 0, \tag{32}$$

since $(K(e_i, e_{i^*}) \circ R)(e_i, e_j)e_i = (K(e_i, e_j) \circ R)(e_{i^*}, e_j)e_i = 0$. Besides, we have by contraction that $K \circ Q = 0$, whence by Theorem 2 it follows that $G = 0$, i.e. we have relation $(*)$ with $k = 0$ or $k = n$. If $k = n$ we get $\lambda^2 = c/4$ and relation (31) immediately gives the contradiction $c = 0$. If $k = 0$, then by relation (32) we immediately get that $c = 0$. Hence, M^n is a totally geodesic hypersurface in $\mathbf{C}^{n+1}$, thus M^n is flat.

Now, assume $Q \circ K = 0$ and suppose that M^n is not flat, i.e. $Q \neq 0$. Then, for distinct indices i and j, we have

$$\text{(33)} \quad (\nu_{ij} + k_{ij})\mu_i = 0, \qquad \text{(34)} \ \nu_{ij}\mu_i = 0, \qquad \text{(35)} \ \bar{\nu}_{ij}\mu_i = 0,$$

since $(Q \circ K)(e_i, e_j)e_i = (Q \circ K)(e_i, e_j)e_{i^*} = (Q \circ K)(e_i, e_{j^*})e_{i^*} = 0$. Adding relations (34) and (35), we get

$$c\mu_i = 0, \tag{36}$$

which since $Q \neq 0$ gives $c = 0$. Hence, relation (34) gives

$$\lambda_i \lambda_j = 0, \ (i \neq j). \tag{37}$$

Thus we get relation $(*)$ with $k = 0$ or $k = 1$. Since $Q \neq 0$, we get $k \neq 0$. Hence, necessarily $k = 1$. Since then $\mu_1 = -2\lambda^2$ and $\mu_n = 0$, relation (33) with $i = 1$, $j = n$ gives the contradiction $\lambda = 0$. Therefore (6) $\Rightarrow$ (8).

Next, assume that $K \circ C = 0$. Let i, j be distinct indices; then, we get from the conditions $(K(e_i, e_j) \circ C)(e_i, e_{i^*})e_i = 0$, $(K(e_i, e_j) \circ C)(e_i, e_{i^*})e_j = 0$, that

$$\nu_{ij}[2(\bar{\nu}_{ii} - \bar{\nu}_{ij}) + \alpha_{ii} - \alpha_{ij}] - k_{ij}(c/2 + \bar{\nu}_{ij}) = 0, \tag{38}$$

$$\nu_{ij}[2(\bar{\nu}_{ii} - \bar{\nu}_{ij}) + \alpha_{ii} - \alpha_{ij}] + k_{ij}(2\bar{\nu}_{ii} - \bar{\nu}_{ij} + \alpha_{ii} - \alpha_{ij} + c/2) = 0 \tag{39}$$

Addition of these formulas implies

$$(2\nu_{ij} + k_{ij})(\lambda_i - \lambda_j) = 0, \tag{40}$$

whereas subtraction yields

$$k_{ij}[3(n-1)c - 2(2n-3)\lambda_i^2 - 2\lambda_j^2] = 0. \tag{41}$$

For mutually distinct indices i and j we obtain

$$\bar{\nu}_{ij}[2(\bar{\nu}_{ii} - \nu_{ii}) + \alpha_{ii} - \alpha_{ij}] - k_{ij}(c/2 + \nu_{ij}) = 0, \tag{42}$$

$$\bar{\nu}_{ij}[2(\bar{\nu}_{ii} - \nu_{ij}) + \alpha_{ii} - \alpha_{ij}] + k_{ij}(2\bar{\nu}_{ii} - \bar{\nu}_{ij} + \alpha_{ii} - \alpha_{ij} + c/2) = 0 \tag{43}$$

since $(K(e_i, e_{j^*}) \circ C)(e_i, e_{i^*})e_j = 0$ and $(K(e_{i^*}, e_j) \circ C)(e_i, e_{i^*})e_j = 0$. Adding (42) and (43) gives

$$(2\bar{\nu}_{ij} + k_{ij})(\lambda_i + \lambda_j) = 0. \tag{44}$$

Multiplying (40) by $\lambda_i + \lambda_j$ and (44) by $\lambda_i - \lambda_j$ and subtracting obtained relations, we get the relation $(*)$. If $k = 0$ then relation (41) immediately gives $c = 0$, thus M^n is a flat hypersurface. If $k = n$, then $\lambda_1 = \cdots = \lambda_n = \lambda > 0$ where $\lambda^2 = c/4$ and relation (41) gives the contradiction $c = 0$. In case $0 < k < n$ we put $i = 1$, $j = n$ in (38). This implies that $\lambda^2 = 3(n+1)c/4n$, $(c \neq 0)$. Substituting this result into (41), we get $c = 0$, which is again a contradiction. This shows that the unique possible case is that M^n is flat and therefore (2) $\Rightarrow$ (8).

Next, assume that $P \circ K = 0$. Let i, j be distinct indices; then we have

$$\lambda_i\lambda_j(2\bar{\nu}_{ii} - \alpha\mu_i) = 0, \ (\alpha = 1/(2n-1)), \tag{45}$$

$$(2\bar{\nu}_{ij} - 2\bar{\nu}_{ii} + k_{ij} - k_{ii})\nu_{ij} - \alpha\mu_i(\bar{\nu}_{ij} + c/2) = 0, \tag{46}$$

$$(2\bar{\nu}_{ij} - 2\bar{\nu}_{ii} + k_{ij} - k_{ii})\nu_{ij} - \alpha\mu_i(\bar{\nu}_{ij} + k_{ij}) + \alpha\mu_j(2\bar{\nu}_{ii} + k_{ii} + c/2) = 0, \tag{47}$$

since $(P(e_i, e_{i^*}) \circ K)(e_i, e_j)e_i = (P(e_i, e_j) \circ K)(e_{i^*}, e_i)e_i = (P(e_i, e_j) \circ K)(e_{i^*}, e_i)e_j = 0$. Interchanging in relation (45) indices i, j and subtracting, we get relation $(*)$. Interchanging in relation (46) indices i, j and subtracting, we get

$$(\lambda_i^2 - \lambda_j^2)[(2n-1)(n-2)\nu_{ij} + (n-1)c/2 + (n-1)\bar{\nu}_{ij}] = 0, \ (i \neq j). \tag{48}$$

Assume that $k = 0$. Then relation (46) gives $c = 0$, thus M^n is flat. Assume that $k = n$. Then relation (45) gives $\lambda^2 = (n-2)c/8(n-1)$ whence as before we have the contradiction $\lambda = 0$. Finally, assume that $0 < k < n$. Then, taking in relation (48) $i = x$ and $j = \alpha$, $(\lambda_x = 0,\ \lambda_\alpha = \lambda > 0)$, we have $c = 0$. Subtracting relations (46) and (47) when we put $i = x$, $j = \alpha$ and $c = 0$, we get $\lambda = 0$, i.e. a contradiction. This completes the proof of (5) $\Rightarrow$ (8).

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