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# A GENERALIZATION OF CHAIN-NET SPACES

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**Abstract.** We describe some classes of topological spaces whose topology can be determined by their $(P_\lambda : \lambda \leq \tau)$-convergent sequences, where $P_\lambda$ is a set of uniform ultrafilters on the cardinal $\lambda$ for every $\lambda \leq \tau$.

**0. Notations.** The usual set theoretic and topological notation is followed. Troughout the paper all the topological spaces are regular (and Hausdorff). The symbol $\omega$ denotes the dicrete space of positive integerrs and $\tau$ and $\lambda$ are infinite cardinals. For any cardinal $\lambda$, $\beta\lambda\backslash\lambda$ will be the set of all free ultrafilters on $\lambda$ (with the discrete topology) and $\mu(\lambda)(= \{p \in \beta\lambda : |A| = \lambda \text{ for each } A \in p\})$ will denote the set of all uniform ultrafilters on $\lambda$ [3]. Let $\tau$ be a cardinal. For every $\lambda \leq \tau$ fix a subset $P_\lambda$ of $\mu(\lambda)$; then $\mathcal{P}$ will denote the collection $\{P_\lambda : \lambda \leq \tau\}$.

**1. Introduction and definitions.** Let $p$ be a free ultrafilter on $\omega$. For a topological space $X$ and a sequence $(x_n : n \in \omega)$ in $X$, a $p$-limit of $(x_n)$, denoted $x = p - \lim x_n$, is a point $x \in X$ such that for every neighborhood $U$ of $x$, $\{n \in \omega : x_n \in U\} \in p$. A space $X$ is called $p$-compact if each sequence in $X$ has a $p$-limit point in $X$.

These notions were introduced by Bernstein in [2]; they play an important role in the theory of products of countably compact spaces (see [5] and [11]).

Kombarov [7] introduced the notions of $P$-compactness and $P$-sequentiality, where $P \subset \beta\omega/\omega$ is a nonempty set of free ultrafilters on $\omega$. In [8] he studied under which conditions $P$-compactness and $P$-sequentiality are preserved by the product operation.

Saks [9] (see also [10]) generalizes the notion of $p$-limit to nets indexed by an arbitrary infinite cardinal $\tau$ as follows: if $p$ is a free ultrafilter on $\tau$ and $(x_\alpha : \alpha \in \tau)$ is a $\tau$-sequence in a space $X$, then a point $x \in X$ is a $p$-limit point of $(x_\alpha)$. $x = p - \lim x_\alpha$, if for every neighborhood $U$ of $x$, $\{\alpha \in \tau\ x_\alpha \in U\} \in p$.

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*Remark.* We say that $(x_\alpha)$ $p$-converges to $x$ whenever $x = p - \lim x_\alpha$. Saks also proves that any topological space is characterized by its (generalized) $p$-limits, in the sense that for any $A \subset X$ one has

$$\bar{A} = A \cup \{x \in X : \ x \text{ is a } p\text{-limit of some } (x_\alpha : \alpha \in \lambda) \text{ for some } \lambda \leq \tau = |X| \\ \text{and some } p \in \beta\lambda \backslash \lambda\}.$$

In light of this fact it is natural to restrict the attention to the class of topological spaces in which the closure of any subset can be obtained by using some special $\tau$ and $p \in \beta\lambda\backslash\lambda$, in particular by using the same (fixed) $p \in \beta\tau\backslash\tau$ for some $\tau$.

Employing the mentioned ideas of Kombarov and Saks we here define the concepts of $(P_\lambda : \lambda \leq \tau)$-radial and $(P_\lambda : \lambda \leq \tau)$-pseudo-radial (or $(P_\lambda : \lambda \leq \tau)$-chain-net) spaces with some variations, where $\tau$ is a cardinal and for every $\lambda \leq \tau$, $P_\lambda$ is a set of uniform ultrafilters on $\lambda$. These definitions are essentially obtained by replacing convergent (transfinite) sequences with $P_\lambda$-convergent sequences in the definitions of radial and pseudo-radial spaces. (Recall that radial and pseudo-radial spaces are precisely the pseudo-open and quotient images, respectively, of orderable spaces [1].)

Let $\lambda$ be a cardinal and $P_\lambda \subset \mu(\lambda)$. We say that a $\lambda$-sequence $(x_\alpha : \alpha \in \lambda)$ in a space $X$

(a) *strongly $P_\lambda$-converges* (to $x$) (abbreviated: $sP_\lambda$-converges) if it $p$-converges to (the same point) $x$ for every $p \in P_\lambda$;

(b) *$P_\lambda$-converges* if it $p$-converges to some point $x(p)$ for every $p \in P_\lambda$;

(c) *weakly $P_\lambda$-converges* ($= wP_\lambda$-converges) (to $x$) if it $p$-converges to a point $x$ for some $p \in P_\lambda$;

(d) *very weakly $P_\lambda$-converges* ($= vwP_\lambda$-converges) if there is a point $x$ with the property that for every neighborhood $U$ of $x$ there exists some $p(U) \in P_\lambda$ such that $\{\alpha \in \lambda : x_\alpha \in U\} \in p(U)$.

Let for each $\lambda \leq \tau$, we have selected a set $P_\lambda \subset \mu(\lambda)$ and let $\mathcal{P} = \{P_\lambda : \lambda \leq \tau\}$. If $A$ is a subset of a space $X$, we define $s\mathcal{P}(A)$ (resp., $w\mathcal{P}(A)$, $vw\mathcal{P}(A)$) to be $A \cup \{x \in X$ : there is some $\lambda \leq \tau$ and a sequence $(x_\alpha : \alpha \in \lambda)$ that $sP_\lambda$-(resp., $wP_\lambda$-, $vwP_\lambda$-) converges to $x\}$.

*Definition* 1.1. Let $\mathcal{P} = \{P_\lambda \subset \mu(\lambda) : \lambda \leq \tau\}$. A space $X$ is called *$s\mathcal{P}$-radial* (resp., *$w\mathcal{P}$-radial*, *$vw\mathcal{P}$-radial*) if $s\mathcal{P}(A) = \bar{A}$ (resp., $w\mathcal{P}(A) = \bar{A}$, $vw\mathcal{P}(A) = \bar{A}$) for every $A \subset X$. $X$ is called $s\mathcal{P}$-pseudo-radial (resp., *$w\mathcal{P}$-pseudo-radial*, *$vw\mathcal{P}$-pseudo--radial*) if $s\mathcal{P}(A) \subset A$ (resp., $w\mathcal{P}(A) \subset A$, $vw\mathcal{P}(A) \subset A$) implies $A$ is closed in $X$.

*Definition* 1.2. Let $\mathcal{P} = \{P_\lambda \subset \mu(\lambda) : \lambda \leq \tau\}$. A space $X$ is said to be *$s\mathcal{P}$-compact* (resp., *$\mathcal{P}$-compact*, *$w\mathcal{P}$-compact*, *$vw\mathcal{P}$-compact*) if for every $\lambda \leq \tau$ each $\lambda$-sequence $(x_\alpha)$ $s\mathcal{P}$-converges (resp., *$\mathcal{P}$-converges*, *$w\mathcal{P}$-converges*, *$vw\mathcal{P}$-converges*) in $X$.

**2. Concerning $\mathcal{P}$-(pseudo) radial spaces.** In what follows $\mathcal{P}$ denotes $\{P_\lambda \subset \mu(\lambda) : \lambda \leq \tau\}$. We start with some trivial facts. It is understood that:

$s\mathcal{P}$-(pseudo) radiality $\Rightarrow$ $w\mathcal{P}$-(pseudo) radiality $\Rightarrow$ $vw\mathcal{P}$-(pseudo) radiality, that every pseudo-radial space is strongly $(\mu(\lambda) : \lambda \leq \tau)$-pseudo-radial (where $\tau$ is the radial tightness [6]) and that all spaces defined by Definition 1.1 have tightness $\leq \tau$.

The following simple result we formulate as a proposition.

Proposition 2.1. *A space $X$ is pseudo-radial if and only if there are regular cardinals $\lambda \leq \tau$ (for some $\tau$) such that $X$ is strongly $(\mu(\lambda) : \lambda \leq \tau)$-pseudo-radial.*

*Proof.* $(\Leftarrow)$ Let $A$ be a non-closed subset of $X$. By hypothesis, there exist a point $x \in \bar{A} \backslash A$, regular $\lambda \leq \tau$ and a $\lambda$-sequence $(x_\alpha)$ in $A$ such that $(x_\alpha)$ $p$-converges to $x$ for each $p \in \mu(\lambda)$. Then, as one can easily check, this $\lambda$-sequence converges to $x$, so that $X$ is pseudoradial.

The proposition above essentially says that we can, and do, assume, without loss of generality, that in Definition 1.1 all cardinals are regular.

The notion of radial order in the class of pseudo-radial spaces was introduced in [6]. We now define a similar ordinal invariant in the class of $s\mathcal{P}$-pseudo-radial spaces. (Of course, in the same way we can define the corresponding concepts in the classes of $w\mathcal{P}$- and $vw\mathcal{P}$-pseudo-radial spaces.)

Let $A$ be a subset of an $s\mathcal{P}$-pseudo-radial space $X$. For every ordinal $\alpha$ we shall define the set $A^\alpha$ as follows; $A^0 = A$, $A^{\alpha+1} = s\mathcal{P}(A^\alpha)$ and if $\alpha$ is a limit ordinal, $A^\alpha = \cup\{A^\beta : \beta < \alpha\}$. Denote by $\mathrm{ir}_{s\mathcal{P}}(X)$ the least ordinal $\alpha$ with the property $A^\alpha = \bar{A}$ for every $A \subset X$. In a similar way as Theorem 2 was proved in [6] one can prove

Theorem 2.2. *A space $X$ is $s\mathcal{P}$-pseudo-radial if and only if $\mathrm{ir}_{s\mathcal{P}}(X)$ exists. In this case $\mathrm{ir}_{s\mathcal{P}}(X) \leq \tau^+$.*

If $X$ is a pseudo-radial space, then we have two ordinal invariants for $X : \mathrm{ir}(X)$ and $\mathrm{ir}_{s\mathcal{P}}(X)$. Obviously, $\mathrm{ir}_{s\mathcal{P}}(X) \leq \mathrm{ir}(X)$. There are simple examples which prove that the inequality $\mathrm{ir}_{s\mathcal{P}}(X) < \mathrm{ir}(X)$ is possible. For instance, the space (of Arens) $M$ in Example 5.1 in [4] is sequential of sequential order 2; but it is strongly $P$-Frechet for suitable $P \subset \beta\omega \backslash \omega$ and so of $P$-sequential order 1.

The following characterizations of $\mathcal{P}$-(pseudo) radial spaces are based on well-known theorems about radial and pseudo-radial spaces, i.e. the use of convergent transfinite sequences should be replaced by the use of $P_\lambda$-convergent sequences. (By a $sP_\lambda$-convergent $\lambda$-sequence one means the union of the sequence $(x_\alpha)$ and its $sP_\lambda$-limit $x$ equipped with the topology in wich every $\{x_\alpha\}$ is open and $(x_\alpha)$ $sP_\lambda$-converges to $x$. In the same way we have $wP_\lambda$ and $vwP_\lambda$ cases.)

Theorem 2.3. *A space $X$ is $s\mathcal{P}$-radial (resp., $w\mathcal{P}$-radial, $vw\mathcal{P}$-radial) if and only if it is a pseudo-open image of a topological sum of $sP_\lambda$- (resp., $wP_\lambda$-, $vwP_\lambda$-) convergent sequences, $\lambda \leq \tau$.*

Theorem 2.4. *A space $X$ is $s\mathcal{P}$-pseudo-radial, (resp., $w\mathcal{P}$-pseudo-radial, $vw\mathcal{P}$-pseudo-radial) if and only if it is a quotient image of a topological sum of $sP_\lambda$- (resp., $wP_\lambda$-, $vwP_\lambda$-) convergent sequences, $\lambda \leq \tau$.*

The following result extends a result of A. Arhangelskii (see Prop. 3 in [1]) about radial spaces to the class of $w\mathcal{P}$-radial spaces.

THEOREM 2.5 *If $X$ is a $w\mathcal{P}$-radial space, then for every $A \subset X$, $|A| \leq 2^{|A|}$, and, in particular, $|X| \leq 2^{d(X)}$.*

*Proof.* Let us suppose first that $|A| \leq \tau$. In this case the set $A$ has at most $2^\tau$ well orderings according to a cardinal $\lambda \leq \tau$. Since ($X$ is Hausdorff and thus) every such transfinite sequence has a unique $wP_\lambda$-limit, the set $\bar{A} = w\mathcal{P}(A)$ has at most $2^\tau$ points, i.e., $|A| \leq 2^{|A|}$ in this case. Now, let $|A| > \tau$. Since $X$ is $w\mathcal{P}$-radial, the tightness of $X$ is $\leq \tau$ and thus $\bar{A} = \cup\{\bar{B} : B \subset A \text{ and } |B| \leq \tau\}$. For each $B \subset A$ with $|B| \leq \tau$ we have $|\bar{B}| \leq 2^\tau$, according to the first part of the proof. Hence one obtains

$$|\bar{A}| \leq |\sum\{\bar{B} : B \subset A \text{ and } |B| \leq \tau\}| \leq 2^{|A|} \cdot 2^\tau = 2^{|A|}.$$

The theorem is proved.

Let $X$ be a $w\mathcal{P}$-pseudo-radial space. We say that a subset $A$ of $X$ is *$w\mathcal{P}$-dense* in $X$ if for every $x \in X$ there is a $\lambda$-sequence, $\lambda \leq \tau$, in $A$ which $wP_\lambda$-converges to $x$. It is natural to define a cardinal function, the *$w\mathcal{P}$-density* $d_{w\mathcal{P}}(X) = \min\{|A| : A$ is $w\mathcal{P}$-dense in $X\}$.

Obviously, $d_{w\mathcal{P}}(X) = d(X)$ for every $w\mathcal{P}$-radial space $X$ and $d(X) \leq d_{w\mathcal{P}}(X)$, for every $w\mathcal{P}$-pseudo-radial space $X$.

THEOREM 2.6. *Let $X$ be a $w\mathcal{P}$-pseudo-radial space. Then*

(i) *For every $A \subset X$, $|w\mathcal{P}(A)| \leq 2^{|A|}$ and in particular, $|X| \leq 2^{d_{w\mathcal{P}}(X)}$.*

(ii) $|X| \leq d_{w\mathcal{P}}(X)^\tau$.

*Proof.* (i) As the proof of the theorem above.

(ii) Let $A$ be a $w\mathcal{P}$-dense subset of $X$ with $|A| = d_{w\mathcal{P}}(X)$. One has $X = w\mathcal{P}(A) = \cup\{w\mathcal{P}(B) : B \subset A \text{ and } |B| \leq \tau\}$ and $|\{B \subset A : |B| \leq \tau\}| \leq |A|^\tau$. According to (i), $|w\mathcal{P}(B)| \leq 2^\tau$, so that we have $|X| \leq |A|^\tau \cdot 2^\tau = |A|^\tau$. The theorem is proved.

THEOREM 2.7. *For every $s\mathcal{P}$-compact space $X$ the following assertion are equivalent:*

(i) *$X$ is $w\mathcal{P}$-pseudo-radial.*

(ii) *Every $s\mathcal{P}$-compact set $A \subset X$ is closed in $X$.*

*Proof.* (i) $\Rightarrow$ (ii) Let $A \subset X$ be $s\mathcal{P}$-compact and suppose that it is not closed. By (i) there exists a $\lambda \leq \tau$, a point $x \in \bar{A}\backslash A$ and a $\lambda$-sequence $(x_\alpha)$ in $A$ such that $(x_\alpha)$ $p$-converges to $x$ for some $p \in P_\lambda$. This means that $(x_\alpha)$ has no $sP_\lambda$-limit points in $A$ which contradicts the assumption that $A$ is $s\mathcal{P}$-compact.

(ii) $\Rightarrow$ (i) Suppose that $A \subset X$ is not closed. By (ii) it cannot be $s\mathcal{P}$-compact and consequently there is $\lambda \leq \tau$ and a $\lambda$-sequence $(x_\alpha)$ in $A$ such that no point in $A$ is $sP_\lambda$-limit of $(x_j)$. But, since $X$ is $s\mathcal{P}$-compact, this $\lambda$-sequence has an $sP_\lambda$-limit point $x$ in $X$, i.e., $x \in \bar{A}\backslash A$. We deduce that $X$ is $w\mathcal{P}$-pseudo-radial. This completes the proof of the theorem.

It is well known that every $w\mathcal{P}$-compact space is initially $\tau$-compact [10]. (Recall that a space $X$ is initially $\tau$-compact if every cover of $X$ of cardinality $\leq \tau$ has a finite subcover.) The example 6.16 in [10] (obtained under GCH) shows that there exists a space $X$ which is initially $\tau$-compact, $\tau > \omega$, but no weakly $(\mu(\lambda) : \lambda \leq \tau)$-compact. But we have the following result.

THEOREM 2.8. *Let $P_\tau \subset \mu(\tau)$. Then every initially $\tau$-compact weakly (strongly) $P_\tau$-pseudo-radial space $X$ is weakly (strongly) $P_\tau$-compact.*

*Proof.* Let $(x_\alpha : \alpha \in \tau)$ be a $\tau$-sequence in $X$. Since $X$ in initially $\tau$-compact this sequence has a complete accumulation point $x \in X$. The set $A = \{x_\alpha : \alpha \in \tau\} \backslash \{x\}$ is non-closed, because $x \in \bar{A} \backslash A$. Since $X$ is weakly (strongly) $P_\lambda$-pseudo--radial there exists a point $y \in \bar{A} \backslash A$ and a $\tau$-sequence $(y_\beta : \beta \in \tau)$ in $A$ such that $y = p - \lim y_\beta$ for some (every) $p \in P_\tau$. In other words, in the sequence $(x_\alpha)$ there is a $\tau$-(sub)sequence $(x_{\alpha\beta} : \beta \in \tau)$ with $y = p - \lim x_{\alpha\beta}$. We wish to prove that $y = p - \lim x_\alpha$. Let $V$ be any neighborhood of $y$. Then $\{\alpha_\beta : x_{\alpha\beta} \in V\} \in p$ and thus $\{\alpha \in \tau : x_\alpha \in V\} \supset \{\alpha_\beta : x_{\alpha\beta} \in V\}$ implies $\{\alpha \in \tau : x_\alpha \in V\} \in p$. So, $X$ is weakly (strongly) $P_\tau$-compact and the theorem is proved.

COROLLARY 2.9. *Let $P \subset \beta\omega\backslash\omega$. Then every countably compact weakly (strongly) $P$-sequential space is weakly (strongly) $P$-compact.*

Now we need the following lemma (which is of independent interest), the proof of which we omit because it is similar to the proofs of Lemmas 1 and 2 of [7].

LEMMA 2.10. *If $X$ is strongly (very weakly) $\mathcal{P}$-pseudo-radial and $Y$ is very weakly (strongly) $\mathcal{P}$-compact, then the projection $\pi_X : X \times Y \to X$ is closed.*

Call a space $X$ locally $s\mathcal{P}$-compact ($w\mathcal{P}$-compact) if for every point $x \in X$ and every neighborhood $U$ of $x$ there exists a neighborhood $V$ of $x$ such that $\bar{V} \subset U$ and $\bar{V}$ is $s\mathcal{P}$-compact ($w\mathcal{P}$-compact).

With this definition we have the following result.

THEOREM 2.11. *If $X$ is a $w\mathcal{P}$-pseudo-radial locally $s\mathcal{P}$-compact (resp., locally $w\mathcal{P}$-compact) space and $Y$ is $w\mathcal{P}$-pseudo-radial (resp., $s\mathcal{P}$-pseudo-radial), then the space $X \times Y$ is $w\mathcal{P}$-pseudo radial.*

*Proof.* Since $Y$ is a $w\mathcal{P}$-pseudo-radial (resp., $s\mathcal{P}$-pseudo-radial) space according to Theorem 2.4, there exists a space $Z$ which is a topological sum of $wP_\lambda$-convergent (resp., $sP_\lambda$-convergent) sequences, $\lambda \leq \tau$, and a quotient mapping $g : Z \to Y$. We claim that the mapping $f = 1_X \times g : X \times Z \to X \times Y$ is quotient. (Note that the proof of this fact is similar to the proof of the well-known theorem of Whitehead which states that if $X$ is locally compact and $g : Y \to Z$ is a quotient mapping, then the mapping $1_X \times g : X \times Y \to X \times Z$ is quotient; see, for example, Engelking's "General Topology".) Let $f^{-1}(W) \subset X \times Z$ be an open set. We shall prove that $W$ is open in $X \times Y$. Choose an arbitrary point $(a, b)$ in $W$ and take any $c \in Z$ with $g(c) = b$. Let $U$ be a neighborhood of $a$ for which $\bar{U}$ is $s\mathcal{P}$-compact (resp., $w\mathcal{P}$-compact) and $\bar{U} \times \{c\} \subset f^{-1}(W)$. One can easily check that $\bar{U} \times g^{-1}(b) \subset f^{-1}(W)$. We are going to prove that the set

$V = \{y \in Y : \bar{U} \times g^{-1}(y) \subset f^{-1}(W)\}$ is open in $Y$, or, since $g$ is quotient, that $g^{-1}(V)$ is open in $Z$. We have $g^{-1}(V) = \{z \in Z : \bar{U} \times \{z\} \subset f^{-1}(W)\}$. This set is open as the complement of the set $\pi_Z(\bar{U} \times Z \setminus f^{-1}(W))$ which is closed because it is the image of the closed set $\bar{U} \times Z \setminus f^{-1}(W)$ under the closed projection $\pi_Z : \bar{U} \times Z \to Z$; $\pi_Z$ is closed according to Lemma 2.10. As $b \in V$ we have $(a, b) \in U \times V \subset W$, i.e. $W$ is open which proves the claim ($f$ is quotient). So, $X \times Y$ is a quotient image of the space $X \times Z$ which is $w\mathcal{P}$-pseudo-radial as can be verified without difficulty. Since $w\mathcal{P}$-pseudo-radiality is preserved by quotient mapping, we have the proof of the theorem.

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