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A NOTE CONCERNING SPECTRAL MULTIPLICITY ONE

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Abstract. In the papers [7, 8] S. Mitrović presents a seemingly remarkable generalization of Cramér's well-known result concerning the spectral multiplicity of a second order stochastic process (see [2]), and then derives some consequences of that generalization. Here, we shall show that (with the exeption of Th. 1. from [8]) the main results of those two papers are false.

1. Introduction. Let $x(t)$, $t \in T = (A, B)$ be a real-valued, purely nondeterministic second order stochastic process, mean-square continuous from the left and with mean-square limits from the right. It is known [2. Th. 2.1] that such a process permits the so-called canonical representation, that is a representation of the form

$$x(t) = \sum_{n=1}^{N} \int_A^t g_n(t,u)\, dz_n(u), \quad t \in T, \tag{1}$$

with the following properties:

(Q_1) $z_1(t), \dots$ are mutually orthogonal processes with orthogonal increments, such that for every n.

$$Ez_n(t) = 0, \;\; Ez_n^2(t) = F_n(t), \;\; t \in T,$$

where $F_n(t)$ is a nondecreasing function continuous from the left;

(Q_2) $g_1(t,u), \dots$ are non-random functions such that

$$Ex^2(t) = \sum_{n=1}^{N} \int_A^t g_n^2(t,u)\, dF_n(u) < \infty, \quad t \in T;$$

(Q_3) $F_1 > F_2 > \cdots;$

(Q_4) $H(x,t) = \sum_{n=1}^{N} \oplus H(z_n,t)$, $t \in T$, where $H(x,t) = \bar{\mathcal{L}}\{x(s),\;\; s \leq t\}$, and $H(z_n,t) = \bar{\mathcal{L}}\{z_n(s),\;\; s \leq t\}$ for every n.

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This representation is canonical in the sense that no other representation of the same form (with properties (Q_1)–(Q_4)) exists for any smaller N. Thus, although the functions $g_n(t,u)$ and the processes $z(t)$ are not uniquely determined by the process $x(t)$, the number N is indeed uniquely determined by the $x(t)$ process. This uniquely determined number N is called the *spectral multiplicity of the $x(t)$ process.*

Each function F_n from $\mathrm{m}(Q_3)$ determines a class R_n of all measures equivalent to F_n. These equivalence classes form a nondecreasing sequence $R_1 > R_2 > \dots$, called *the spectral type of the $x(t)$ process*; the spectral type is uniquely determined by the $x(t)$ process.

Cramer's result mentioned above provides a way to represent a second order process $x(t)$ by means of N (where N is uniquely determined by $x(t)$) mutually orthogonal processes with orthogonal increments. It is natural to ask whether it is possible to give sufficient conditions under which such a representation would involve only one process with orthogonal increments (that is sufficient conditions for $N = 1$). In [2, Th. 5.1], Cramér gives such a set of sufficient conditions; he proves the following result: If X is the class of all processes $x(t)$ admitting canonical representations of of the form (1.1) in which each term on the right hand side satisfies the conditions:

(R_1) $g_n(t,u)$ and $\partial g_n(t,u)/\partial t$ are bounded and continuous for $u, t \in T,\ u \le t$;

(R_2) $g_n(t,t) = 1,\ t \in T$;

(R_3) $F_n(u) = Ez_n^2(u)$ is absolutely continuous and not identically constant, with $f_n(u) = dF_n(u)/du$ having at most finitely many discintinuities in any finite subinterval of T;

then X consists of processes with the spectral multiplicity $N = 1$. It is this result that S. Mitrović attempts to generalize in [7]. There, in the first of two theorems, S. Mitrović states that, for a second order process $x(t)$ to have the spectral multiplicity $N = 1$, it is sufficient that each term in its canonical representation (1.1) satisfies conditions (R_1), (R_3) listed above and

(R_2^*) $g_n(t,t) = 0,\ t \in T$.

Thus, the sufficient conditions of S. Mitrović differ from those of Cramér only in (R_2), where the contrast is substantial.

The other results in [7] and [8] are derived from the first theorem from [7], quoted in the previous paragraph; thus, their validity depend entirely on the latter theorem. The only result that is independent from the first theorem in [7] is Th. 1 in [8]; this theorem, however, is proved as Example 11 in [3].

2. Fallacy of the main result from [7]. In proving the first theorem in [7], S. Mitrović repeats word-by-word the argument used by Cramér in the proof of Th. 5.1 in [2], seemingly not being aware that the specific statement which are true for integral equations of Volterra of the second type appearing in [2, Th. 5.1] are not necessarily true for integral equations of Volterra of the first type appearing in [7]; aspects in which those two types of integral equations differ concern the existence and uniqueness of their solutions, that is exactly what S. Mitrović uses (for a brief discussion of these integral equations, see e.g. [5]). Let us clarify this. The integral equations that S. Mitrović considers (those that the proof of the first theorem in [7] depends upon) are

$$\int_A^s \partial g_n(s,u)/\partial s h_n(u) f_n(u) du = (-1)^{n+1}, \;\; s \in (A,t], \;\; t \in T, \;\; n = 1,2. \tag{2.1}$$

where $g_n(s,u)$ and $f_n(u)$, $n = 1,2$, are from the canonical representation of $x(s)$, and $h_1(u)$, $h_2(u)$ are functions to be determined. S. Mitrović claims that, being Volterra equations of the first type, the equations (2.1) have unique continuous solutions that are almost everywhere (with respect to the respective measures $F_1(u)$, and $F_2(u)$) different from zero.

It is easy to see, however, that functions $F_n(u) = u$ and

$$g_n(s,u) = \begin{cases} s-u, & u \le s \\ 0, & u > s \end{cases}, \quad n = 1,2,$$

satisfy conditions (R_1), (R_2^*) and (R_3), but the equations (2.1) do not have continuous solutions.

Although this simple example shows that the proof of the main theorem in [7] is false, it still does not provide a clear evidence that the main result itself is false. Such an evidence, however, can be found in [1, Th. 3] and [9, pp. 8–9], where in each case a process of infinite spectral multiplicity, satisfying (R_1), (R_2^*), (R_3), is constructed. To simplify the discussion and make our point completely clear, we shall construct a process that satisfies conditions (R_1), (R_2^*) and (R_3) of S. Mitrović, but has spectral multiplicity $N = 2$; our procedure will be a combination of those of [1] and [9].

Example. Let $T = (0,1)$. A sequence $C_1, \dots, C_n, \dots$ of unions of Cantor sets will be constructed in the following way:

C_1 is a Cantor set from T, such that $m(C_1) = 1/2$ (here and thereafter, m will denote the Lebesgue measure, and A^c will denote the complement of the set A); $T \cap C_1^c$ sonsists of countably many open intervals. C_2 is the union of Cantor sets obtained on all the open intervals of $T \cap C_1^c$; the set C_2 is constructed so that the Lebesgue measure of each of its Cantor components is equal to 1/2 lenght of the interval in wich it is contained, so that $m(C_2) = 1/2^2$. $T \cap C_1^c \cap C_2^c$ consists of countably many open intervals. Sets $C_3, C_4, \dots, C_n, \dots$ are constructed in an analogous way, namely C_n is the union of Cantor sets obtained on all the open intervals of $C \cap C_1^c \cap \cdots \cap C_{n-1}^c$; the set C_n is constructed so that the Lebesgue measure of each of its Cantor components is equal to 1/2 lenght of the interval in wich it is contained, so that $m(C_n) = 1/2^n$.

Let us now define sets A_1 and A_2 in the following way:

$$A_1 = \bigcup_{n=1}^{\infty} C_{2n-1}, \;\; A_2 = \bigcup_{n=1}^{\infty} C_{2n}.$$

It is clear that $A_1 \cap A_2 = \emptyset$ and $m(A_1) + m(A_2) = 1 = m(T)$. In the argument that will follow, we shall use the fact that

$$m(A_k \cap (a,b)) > 0, \ k = 1,2, \tag{2.2}$$

for any choice of the numbers a, b such that $0 \leq a < b \leq 1$. To prove this, let us take an arbitrary subinterval $i = (a,b)$ of T. It is easy to see that, for some n, i contains an end-point of at least one of the open intervals from $T \cap C_1^c \cap \cdots \cap C_n^c$ (indeed, the converse would mean that, for each n, i is contained in one of thoos intervals; since the lenght of i is constant and $m(T \cap C_1^c \cap \cdots \cap C_n^c) \to 0, \ n \to \infty$, this is impossible). Let I be one of the end-points of a subinterval from $T \cap C_1^c \cap \cdots \cap C_n^c$ that is in i; we can assume that, for instance, I is the left end-point of the interval $(I, J) \subset T \cap C_1^c \cap \cdots \cap C_n^c$. Let $(I, I^*) = (I, J) \cap i$. After finitely many steps, say k, we shall find an interval $(I_1, I_2) \subset T \cap C_1^c \cap \cdots \cap C_{n+k}^c$ such that $(I_1, I_2) \subset (I, I^*)$. In the next step, a Cantor set from (I_1, I_2) will be used to make one of the sets A_1 and A_2, say A_1. Consequently, we have

$$m(A_1 \cap i) > m(A_1 \cap (I_1, I_2)) > (I_2 - I_1)/2 > 0.$$

By advancing the same argument for one more step, we obtain

$$m(A_2 \cap i) > m(A_2 \cap (I_1, I_2)) > (I_2 - I_1)/2^2 > 0.$$

In this way, (2.2) it proved.

Let us now denote by $\alpha_n(u)$ the indicator-function of the set A_n, $n = 1,2$, and let us define functions $g_1(t,u)$ and $g_2(t,u)$ in the following way:

$$g_n(t,u) = \begin{cases} \int_u^t (t-\nu)\alpha_n(\nu)\, d\nu, & u < t \\ 0, & u \geq t \end{cases}, \quad n = 1,2.$$

Moreover, let $z_1(u)$ and $z_2(u)$ be mutually orthogonal real-valued random processes with orthogonal increments; it will be assumed that

$$Ez_n(u) = 0, \ \ F_n(u) = Ez_n^2(u), \ \ u \in T, \ \ n = 1,2.$$

It is clear that the above functions $g_n(t,u)$ and $F_n(u)$, $n = 1,2$, satisfy the conditions (R_1), (R_2^*), (R_3).

Let us process $x(t)$ be defined by

$$x(t) = \sum_{n=1}^{2} \int_0^t g_n(t,u)\, dz_n(u), \ \ t \in T.$$

We shall show that the spectral multiplicity of $x(t)$ is $N = 2$.

It is easy to check that the conditions (Q_1), (Q_2) and (Q_3) are satisfied. To prove that $N = 2$ it is sufficient (see Lemma 3.1 from [2]) to show that (Q_4) is satisfied as well. A sufficient condition for (Q_4) is

$$z_n(t) \in H(x,t), \ \ t \in T, \ \ n = 1,2. \tag{2.3}$$

It is this last relation that we shall prove.

If we show that

$$z_n(t) = d^2x(t)/dt^2,\ t \in T,\ n = 1, 2,$$

then, because $d^2x(t)/dt^2 \in H(x,t)$, it will mean that (2.3) is satisfied and, ultimately, that $N = 2$.

By writing $x(t)$ in the form

$$x(t) = \sum_{n=1}^{2} \int_A^t (t-\nu)\alpha_n(\nu) z_n(\nu)\, d\nu,\ t \in T,$$

and using the definition of the derivative, it is not difficult to obtain

$$dx(t)/dt = \sum_{n=1}^{2} \int_A^t \alpha_n(\nu) z_n(\nu)\, d\nu,\ t \in T.$$

To find $d^2x(t)/dt^2$, one can proceed as follows. The definition of the derivative gives

$$\begin{aligned} d^2x(t)/dt^2 = l.i.m._{h\to 0}\frac{1}{h}\Bigg[&\sum_{n=1}^{2}\int_t^{t+h} \alpha_n(\nu)\,(z_n(\nu) - z_n(t)) d\nu \\ &+ \sum_{n=1}^{2} z_n(t) \int_t^{t+h} \alpha_i(\nu) d\nu\Bigg]. \end{aligned} \tag{2.4}$$

Since each of the sets A_1, A_2 has either zero or unit metric density at almost any $t \in (A, B)$ (see [6, § 140]), we have

$$l.i.m._{h\to 0}\frac{1}{h}\int_t^{t+h} \alpha_n(\nu) d\nu = \begin{cases} 0, & \text{almost everywhere on } A_n^c, \\ 1, & \text{almost everywhere on } A_n, \end{cases}$$

and thus

$$l.i.m._{h\to 0}\frac{1}{h}\sum_{n=1}^{2} z_n(t)\int_t^{t+h} \alpha_n(\nu) d\nu = \sum_{n=1}^{2} z_n(t)\alpha_n(t). \tag{2.5}$$

Without difficulties, it can be shown that

$$E\left|\sum_{n=1}^{2}\int_t^{t+h} \alpha_n(\nu)(z_n(\nu) - z_n(t)) d\nu\right|^2 \le 2h^3,$$

which implies that

$$l.i.m._{h\to 0}\frac{1}{h}\sum_{n=1}^{2}\int_t^{t+h} \alpha_n(\nu)(z_n(\nu) - z_n(t)) d\nu = 0. \tag{2.6}$$

Equations (2.5) and (2.6), together with (2.4), give

$$d^2x(t)/dt^2 = \sum_{n=1}^{2} z_n(t)\alpha(t) \quad \text{a.e. on } T.$$

Thus,

$$d^2x(t)/dt^2 = z_n(t) \quad \text{a.e. on } A_n. \tag{2.7}$$

Since $d^2x(t)/dt^2 \in H(x,t)$, (2.7) means that

$$z_n(t) \in H(x,t) \quad \text{a.e. on } A_n.$$

However, since process $z_n(t)$, $n = 1, 2$, is continuous to the left and A_n, $n = 1, 2$, has nonzero measure in every interval from T, it follows that (2.3) is satisfied. Consequently, it is proved that process $x(t)$ has spectral multiplicity $N = 2$, in spite of the fact that conditions (R_1), (R_2^*) and (R_3) are satisfied.

Being dependent on the first theorem from [7], the validity of the remaining result from [7] as well as those from [8] (with the exception of already mentioned Th. 1) is in doubt at least in their present form.

At the end, it should be pointed out that, additionally, some statements and some proofs from both [7] and [8] are too vague and incomplete. For instance, the statement (given at the beginning of [8]) that, if a process $x(t) = \int_A^t g(t,u)\,dz(u)$, $t \in T$, satisfies (R_1), (R_2^*), (R_3), then its spectral multiplicity is equal to one, does not follow from [7], as S. Mitrović asserts. Also, since the continuity of a function of two variables is not a consequence of its continuity with respect to each of the variables separately (see [4, Ch. 9, Ex. 1]), S. Mitrović does not derive a full proof of the Lemma in [8]. Finally, it should be mentioned that, apparently because of the inconsistency in using the terminology, Th. 2 and Th. 3 from [8] represent but one statement.

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