PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 44 (58), 1988, pp. 127–136

SUFFICIENT CONDITIONS FOR UNIFORM CONVERGENCE OF A CLASS OF SPLINE DIFFERENCE SCHEMES FOR SINGULARLY PERTURBED PROBLEMS

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Abstract. A family of spline difference schemes for singularly perturbed boundary value problems is derived. The schemes have the first order of the uniform convergence. Numerical results are presented.

1. Introduction. The cubuc spline difference scheme for the problem

$$\begin{aligned} &Ly \equiv \varepsilon y'' + p(x)y' = f(x), \quad 0 < x < 1 \\ &y(0) = \alpha, \quad y(1)\beta \\ &p(x) > 0, \end{aligned} \tag{1.1}$$

when $\varepsilon = 1$ was derived by Il'in [6]. In order to avoid the difficulties relative to the cell Raynolds number the exponentialy fitted factor affecting the highest derivative was used by Surla [8]. So, the uniformly convergent spline difference scheme was obtained. In both papers the spline function $\nu(x)$ from $C^2[0,1]$ was used.

Our object is to weaken the continuity conditions by taking $\nu(x) \in C_1[0,1]$ and try to get a scheme with better accuracy and simpler form. In such way the schemes in Section 2 are obtained. These schemes belong to the family of exponentialy fitted schemes which are analysed in detail by Doolan, Miller, Schilders [3] and Farrell [4]. One of them is well-known Allen-Southwell-Il'in scheme.

Using the approach of Kellog, Tsan [7] a complete error analysis is given in Section 3. Some schemes wich belong to this family are analysed in [9].

Some new schemes and numerical results are presented in Section 4.

2. Derivation of the schemes. Let n be a positive integer and define the uniform mesh $\{x_j\}$ by $x_j = jh$, $j = 0, \dots, n+1$, where mesh lenght $h = 1/(n+1)$.

AMS Subject Classification (1980): Primary 65L10, 65D07.

The spline $\nu(x)$ on each interval $I_j = [x_j, x_{j+1}]$, $j = 0, \dots, n$ has the following form:

$$\nu_j(x) = \nu_j^{(0)} + (x - x_j)\nu_j^{(1)} + (x - x_j)^2 \nu_j^{(2)}/2 + (x - x_j)^3 \nu_j^{(3)}/6, \quad x \in I_j.$$

$\nu_j^{(0)}$ is the approximate value (to be determined) for $y_j = y(x_j)$.

The unknown coefficients $\nu_j^{(K)}$, $K = 0, 1, 2, 3$, $j = 0, \dots, n$ will be determined from the following conditions:

$\nu_j(x)$ satisfies, at the grid points x_j and x_{j+1}, the following comparison problem to the problem (1) with artifical viscosity $\tilde{\sigma}_j(x)$ ($\sigma(x)$ will be determined):

$$\tilde{L}\tilde{y}(x) \equiv \tilde{\sigma}(x)\tilde{y}''(x) + \tilde{p}(x)\tilde{y}'(x) = \tilde{f}(x) \quad \tilde{y}(0) = \alpha, \quad \tilde{y}(1) = \beta, \quad x \in [0, 1],$$

$\tilde{\sigma}(x)$, $\tilde{p}(x)$ and $\tilde{f}(x)$ are approximations to $\sigma(x)$, $p(x)$ and $f(x)$ respectively,

$$\alpha_0(0) = \nu, \quad \nu_n(1) = \beta, \quad \nu(x) \in C^1[0, 1].$$

From the conditions above we can form the system of $4n + 4$ equations with $4n + 4$ unknowns:

$$\begin{aligned}
&\tilde{L}\nu_j(x)_{x=x_j} = \tilde{f}_j(x)_{x=x_j}, && j = 0, \dots, n, \\
&\tilde{L}\nu_j(x)_{x=x_{j+1}} = \tilde{f}_j(x)_{x=x_{j+1}}, && j = 0, \dots, n, \\
&\nu_j(x)_{x=x_{j+1}} = \nu_{j+1}(x)_{x=x_{j+1}}, && j = 0, \dots, n-1, \\
&\nu_j'(x)_{x=x_{j+1}} = \nu_{j+1}'(x)_{x=x_{j+1}}, && j = 0, \dots, n-1, \\
&\nu_0(0) = \alpha, \quad \nu_n(1) = \beta.
\end{aligned} \tag{2.1}$$

When $\tilde{\sigma}(x)$, $\tilde{p}(x)$ and $\tilde{f}(x)$ are piecewise constant approximations to $\sigma(x)$, $p(x)$ and $f(x)$ respectively ($\tilde{p}(x) = \tilde{p}_{j-1}$ for $x \in I_{j-1}$, etc.), then the system (2.1) on the interval I_{j-1} has the following form:

$$\begin{aligned}
&\tilde{\sigma}_{j-1}\nu_{j-1}^{(2)} + \tilde{p}_{j-1}\nu_{j-1}^{(j)} = \tilde{f}_{j-1}, \\
&\tilde{\sigma}_{j-1}(\nu_{j-1}^{(2)} + h\nu_{j-1}^{(3)}) + \tilde{p}_{j-1}(\nu_{j-1}^{(1)} + h\nu_{j-1}^{(2)} + h^2\nu_{j-1}^{(3)}/2) = \tilde{f}_{j-1}, \\
&\nu_{j-1}^{(0)} + h\nu_{j-1}^{(1)} + h^2\nu_{j-1}^{(2)}/2 + h^3\nu_{j-1}^{(3)}/6 = \nu^{(0)}, \\
&\nu_{j-1}^{(1)} + h\nu_{j-1}^{(2)}) + h^2\nu_{j-1}^{(3)}/2 = \nu_j^{(1)}.
\end{aligned}$$

By expressing $\nu_{j-1}^{(2)}$ from the first equation and $\nu_{j-1}^{(3)}$ from the second equation, the third and the fourth equations have the following form:

$$\nu_j^{(0)} = \nu_{j-1}^{(0)} + \tilde{\gamma}_{j-1}\nu_{j-1}^{(1)} + \tilde{S}_{j-1}\tilde{f}_{j-1}, \tag{2.2}$$

$$\nu_j^{(1)} = \tilde{A}_{j-1}\nu_{j-1}^{(1)} + 2h\tilde{f}_{j-1}/(2\tilde{\sigma}_{j-1} + h\tilde{p}_{j-1}) \tag{2.3}$$

where

$$\tilde{A}_{j-1} = \frac{2\tilde{\sigma}_{j-1} - \tilde{p}_{j-1}h}{2\tilde{\sigma}_{j-1} + \tilde{p}_{j-1}h}, \quad \tilde{S}_{j-1} = \frac{h^2(6\tilde{\sigma}_{j-1} + \tilde{p}_{j-1}h)}{6\tilde{\sigma}_{j-1}(2\tilde{\sigma}_{j-1} + \tilde{p}_{j-1}h)}, \quad \tilde{\gamma}_{j-1} = \frac{h(12\tilde{\sigma}_{j-1}^2 - \tilde{p}_{j-1}^2h^2)}{6\tilde{\sigma}_{j-1}(2\tilde{\sigma}_{j-1} + \tilde{p}_{j-1})h}.$$

On the interval I_j we have:

$$\nu_{j+1}^{(0)} = \nu_j^{(0)} + \tilde{\gamma}\tilde{\nu}_j^{(1)} + \tilde{S}\tilde{f}_j, \tag{2.4}$$

$$\nu_{j+1}^{(1)} = \tilde{A}_j \nu_j^{(1)} + 2hf_j/(2\tilde{\sigma}_j + h\tilde{p}_j) \tag{2.5}$$

From (2.2) we get

$$\nu_{j-1}^{(1)} = (\nu_j^{(0)} - \nu_{j-1}^{(0)} - \tilde{S}_{j-1}\tilde{f}_{j-1})/\tilde{\gamma}_{j-1} \tag{2.6}$$

and from (2.4)

$$\nu_j^{(1)} = (\nu_{j+1}^{(0)} - \nu^{(0)} - \tilde{S}_j\tilde{f}_j)/\tilde{\gamma}_j \tag{2.7}$$

By putting (2.7) and (2.6) into equation (2.3) we get the following difference scheme:

$$\frac{\tilde{A}_{j-1}}{\tilde{\gamma}_{j-1}} \cdot \nu_{j-1}^{(0)} - \left(\frac{\tilde{A}_{j-1}}{\tilde{\gamma}_{j-1}} + \frac{1}{\tilde{\gamma}_j}\right)\nu_j^{(0)} + \frac{1}{\tilde{\gamma}_j}\nu_{j+1}^{(0)} = \frac{\tilde{S}_j}{\tilde{\gamma}_j}\tilde{f}_j + \left(\frac{2h}{2\tilde{\sigma}_{j-1} + h\tilde{p}_{j-1}} - \frac{\tilde{A}_{j-1}\tilde{S}_{j-1}}{\tilde{\gamma}_{j-1}}\right)\tilde{f}_{j-1}. \tag{2.8}$$

We can see that scheme (2.8) is a member of the family of implicit difference schemes wich have the form:

$$\tilde{R}\nu_j^{(0)} = \tilde{Q}\tilde{f}_j, \quad j = 1, 2, \ldots, n \tag{2.9}$$

where

$$\tilde{R}\nu_j^{(0)} \equiv \tilde{r}_j^- \nu_{j-1}^{(0)} + \tilde{r}_j^c \nu_j^{(0)} + \tilde{r}_j^+ \nu_{j+1}^{(0)} \qquad \tilde{Q}\tilde{f}_j \equiv \tilde{q}_j\tilde{f}_{j-1} + q_j^c f_j + \tilde{q}_j^+\tilde{f}_{j+1}.$$

The local truncation error $\tau_j(g(x))$ of scheme (2.9), for an arbitrary, sufficiently smooth, function is defined by

$$\tau_j(g) = \tilde{R}g(x_j) - \tilde{Q}(\tilde{L}g(x_j)) \tag{2.10}$$

The following lemma from Berger et al. [2] gives properties of the exact solution of (1.1) wich we take into account in derivation of the fitting factor $\sigma(x)$.

Lemma 2.1 *Let $p(x) \in C^3[0,1]$. Then the solution of (1) can be writen in the form $y(x) = u(x) + W(x)$ where*

$$u(x) = \frac{\varepsilon y'(0)}{p(0)} \cdot \exp\left(\frac{-p(0)x}{\varepsilon}\right) \tag{2.11}$$

$$|W_{(x)}^{(i)}| \leq M(1 + \varepsilon^{-i+1}\exp(-2\delta x_j/\varepsilon)), \quad i = 0, 1, 2, 3, 4.$$

M and δ are constants independent of ε.

The fitting factor $\sigma(x)$ we shall find from the condition that truncation error (2.10) for function $u(x)$ (2.11), when $p(x) = p = \text{const}$, will be equal to zero. In that way the error influenced by the boundary layer function is diminished.

LEMMA 2.2. *Let* $p(x) = p = \text{const}$ *in equation,* (1.1) *then* $\tau_j(u) = 0$ *for* $\tilde{\sigma} = \frac{hp}{2}\operatorname{cth}\frac{hp}{2\varepsilon}$.

Proof. For $p(x) = p = \text{const}$ we have: $\frac{\tilde{r}_{j-}}{\tilde{r}_{j+}} = \frac{\tilde{r}^-}{\tilde{r}^+} = \tilde{A} = \frac{\tilde{\omega}-1}{\tilde{\omega}+1} = e^{hp/\varepsilon}$ where $\tilde{\omega} = \operatorname{cth}(hp/2\varepsilon)$. As $Lu(x) = 0$ it easy to see that, $\tau_j(u) = 0$.

For $p(x) \neq \text{const}$, $\tilde{\sigma}_j(x)$ is defined in the following form:

$$\tilde{\sigma}_j = h\tilde{p}_j\tilde{\omega}_j/2, \;\; x \in I_j, \;\; \text{where } \tilde{\omega}_j = \operatorname{cth}(h\tilde{p}_j/2\varepsilon) \tag{2.12}$$

Now, the coefficients of scheme (2.8) have the following form:

$$\begin{gathered}
\tilde{r}_j^- = \frac{3\tilde{\omega}_{j-1}(\tilde{\omega}_{j-1}-1)}{h(3\tilde{\omega}_{j-1}^2-1)}, \quad \tilde{r}_j^+ = \frac{3\tilde{\omega}_j(\tilde{\omega}_j+1)}{h(3\tilde{\omega}_j^2-1)}, \\
\tilde{r}_j^c = -\tilde{r}_j^- - \tilde{r}_j^+, \quad \tilde{q}_j^+ = 0 \\
\tilde{q}_j^- = \frac{3\tilde{\omega}_{j-1}-1}{2\tilde{p}_{j-1}(3\tilde{\omega}_{j-1}^2-1)}, \quad \tilde{q}_j^c = \frac{3\tilde{\omega}_j+1}{\tilde{p}_j(3\tilde{\omega}_j^2-1)}
\end{gathered} \tag{2.13}$$

The choice of approximation to $p(x)$ and $f(x)$ determines the particular scheme.

Choosing $\tilde{p}_j = \tilde{p}_{j-1} = p_j$, $\tilde{f}_j = \tilde{f}_{j-1} = f_j$ scheme (2.13) becomes the scheme of the form $R\nu_j^{(0)} = Qf_j$ where

$$\begin{gathered}
r_j^- = p_j(\omega_j-1)/2h, \;\; r_j^+ = p_j(\omega_j+1)/2h, \;\; r_j^c = -r_j^- - r_j^+, \\
q_j^- = q_j^+ = 0, \;\; q_j^c = 1, \;\; \omega_j = \operatorname{cth}(hp_j/2\varepsilon).
\end{gathered} \tag{2.14}$$

This is precisely the Allen-Southwell-Il'in finite difference scheme. It was proved by Il'in [5], Kellog & Tsan [7] that this method has the first order of the uniform convergence at the nodes. So, the spline difference method (2.14) also must be uniformly convergent of the first order at the nodes.

Choosing $\tilde{p}_{j-1} = (p_{j-1}+p_j)/2$, $\tilde{p}_j = (p_j+p_{j+1})/2$, $\tilde{f}_{j-1} = (f_{j-1}+f_j)/2$, $\tilde{f}_j = (f_j+f_{j+1})/2$ the corresponding implicit difference scheme has the coefficients:

$$\begin{gathered}
r_j^- = \frac{3\omega_{j-1}(\omega_{j-1}-1)}{h(3\omega_{j-1}^2-1)}, \;\; r_j^+ = \frac{3\omega_j(\omega_j+1)}{h(3\omega_j^2-1)}, \;\; r_j^c = -r_j^- - r_j^+, \\
q_j^- = \frac{3\omega_{j-1}-1}{2\tilde{p}_{j-1}(3\omega_{j-1}^2-1)}, \;\; q_j^+ = \frac{3\omega_j+1}{2\tilde{p}_j(3\omega_j^2-1)}, \;\; q_j^c = q_j^- + q_j^+.
\end{gathered} \tag{2.15}$$

This scheme is analysed in [9].

For the family (2.13) the following theorem holds.

THEOREM 2.1. *Let* $\{\nu_j^{(0)}\}$, $j = 0, 1, \ldots, n+1$ *be the approximation to the solution* $y(x_j)$ *of* (1.1) *obtained using* (2.13). *Let* $p(x),\ f(x) \in C^3[0,1]$,

$$\tilde{p}_j = p(x_j) + C_0 h, \quad \tilde{f}_j = f(x_j) + C_1 h; \tag{2.16}$$

then

$$|\nu_j^{(0)} - y(x_j)| \le Mh \tag{2.17}$$

where C_0, C_1 *and* M *are constants independent of* ε *and mesh length* h.

3. Proof of the uniform convergence. In this section we will prove Theorem 2.1. The proof is based on the comparison functions method developed by Kellogg & Tsan [7] and Berger et al. [2]. This method uses the following two lemmas:

LEMMA 3.1. *Let* $\{V_j\}$ *be a set of values at the grid points* $\{x_j\}$, $j = 0, \ldots, n+1$ *satisfying* $V_0 \le 0$, $V_{n+1} \le 0$ *and* $\tilde{R}V_j \ge 0$, $j = 0, \ldots, n$. *Then* $V_j \le 0$ *for* $j = 0, 1, \ldots, n+1$.

LEMMA 3.2. *If* $K_1(h,\varepsilon) \ge 0$ *and* $K_2(h,\varepsilon) \ge 0$ *are functions that satisfy:* $R(K_1(h,\varepsilon)\varphi_j + K_2(h,\varepsilon)\psi_j) \ge R(\pm z_j) = \pm\tau_j(y)$ *where* $z_j = y_j - \nu_j^{(0)}$; *then* $|z_j| \le K_1(h,\varepsilon)|\varphi_j| + K_2(h,\varepsilon)|\psi_j|$.

As in Berger et al. [2] we use two comparison functions $\varphi_j = -2 + x_j$ and $\psi_j = -\exp(\beta x_j/\varepsilon) = -(\mu(\beta))^j$, where $\mu(\beta) = \exp(-\beta h/a)$, $\beta > 0$ will be chosen appropriately.

Troughout the paper let δ, M, M_1, ..., denote positive constants that may take different values in different formulas, but that are always independent of ε and h.

LEMMA 3.3. *There are constants* M_1 *and* M_2 *such that for* $h \le M_1$, $0 < \beta < M_2$ *and* $j = 1, \ldots, n$ *the following holds*

$$\tilde{R}\varphi_j \ge Mh/\varepsilon, \qquad h \le \varepsilon \tag{3.1}$$

$$\tilde{R}\varphi_j \ge M, \qquad \varepsilon \le h \tag{3.2}$$

$$\tilde{R}\psi_j \ge M\mu^j(\beta)h/\varepsilon^2, \qquad h \le \varepsilon \tag{3.3}$$

$$\tilde{R}\psi_j \ge M\mu^j(\beta)/h, \qquad \varepsilon \le h \tag{3.4}$$

$$\tilde{R}\psi_j/\mu(\beta) \ge M\mu^{j-1}(\beta)/h, \qquad \varepsilon \le h. \tag{3.5}$$

The proof of the Lemma 3.3 is similar to the proof of the coresponding lemma in [9].

Now we will consider the truncation error of the schemes (2.13) under the conditions (2.16).

LEMMA 3.4 *Let* (2.16) *hold; then for the truncation error* (2.10) *for family of the scheme* (2.13) *we get the following estimates:*

(3.6) $\quad |\tau_j(y)| \le M(h + \exp(-\delta x_{j-1}/\varepsilon)),\ \ j = 1, \ldots, n \ \textit{for}\ \varepsilon \le h,$

(3.7) $\quad |\tau_j(y)| \le M(h^2/\varepsilon + h^2\varepsilon^{-2}\exp(-\delta x_j/\varepsilon)),\ \ j = 1, \ldots, n \ \textit{for}\ h \le \varepsilon,$

Proof. From (2.10) we have

$$\tau_j(y) = \tilde{R}(y_j - \nu_j^{(0)}) = \tilde{R}y_j - \tilde{Q}(Ly_j + O(h)) = \tilde{\tau}_j(y) + N$$

where

$$|N| \le Mh^2/\max(h, \varepsilon), \quad \tilde{\tau}_j(y) = \tilde{R}y_j - \tilde{Q}(Ly_j).$$

For $y(x)$ sufficiently smooth, the standard Taylor development of $\tilde{\tau}_j$ for fixed ε has the form:

$$\begin{aligned}\tilde{\tau}_j(y) = T_0 y_j + T_1 y_j' + T_2 y_j'' - \tilde{r}_j^- R_2(x_{j-1}, x_j, y) + \tilde{r}_j^+ R_2(x_j, x_{j+1}, y)\\ + \varepsilon\tilde{q}_j^- R_0(x_{j-1}, x_j, y'') - p_{j-1}\tilde{q}_j^- R_1(x_{j-1}, x_j, y'),\end{aligned}$$

where

$$Rn(a, b, g) = g^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!} = \frac{1}{n!}\int_a^b (b-s)^n g^{(n+1)}(s)ds, \quad \xi \in (a, b).$$

When it is clear from the context, the subscripts in $\tilde{r}_j^-, \ldots, \tilde{q}_j^+$ will be omitted.

Since $\tilde{\tau}_j(y) = \tilde{\tau}_j(u) + \tilde{\tau}_j(W)$, (see Lemma 2.1), we will estimate $\tilde{\tau}_j(u)$ and $\tilde{\tau}_j(W)$ separately.

Since $T_0 = T_1 = 0$ we shall estimate T_2 and the remainder terms.

$$T_2 = h^2(\tilde{r}^- + \tilde{r}^+)/2 + p_{j-1}h\tilde{q}^- - \varepsilon(\tilde{q}^- + q^c). \tag{3.8}$$

Since $1 \le \tilde{\omega} \le M\max(\varepsilon/h, 1)$ from (3.8) and Lemma 2.1 we get

$$|T_2 W''| \le Mh, \quad \text{for } \varepsilon \le h. \tag{3.9}$$

In the case $\varepsilon \le h$ consider the remainder term $Y_j \equiv p_{i-1}\tilde{q}^- R_1(x_{j-1}, x_j, W')$ (the others being similar).

$$\begin{aligned}|Y_j| &= |\tilde{q}^- p_{j-1}(R_0(x_{j-1}, x_j, W') + hW''(x_j))|\\ &\le M\int_{x_{j-1}}^{x_j}(1 + \varepsilon^{-1}\exp(-\delta s/\varepsilon))ds + h(1 + \varepsilon^{-1}\exp(-\delta x_j/\varepsilon)).\end{aligned}$$

So

$$|Y_j| \le M(h + \exp(-\delta x_{j-1}/\varepsilon)), \quad j - 1, \ldots, n \ \text{ for } \varepsilon \le h. \tag{3.10}$$

From (3.9) and (3.10) we get

$$|\tilde{\tau}_j(W)| \le M(h + \exp(-\delta x_{j-1}/\varepsilon)), \text{ for } \varepsilon \le h. \tag{3.11}$$

Let us consider now the case $h \le \varepsilon$. For $p(x) = \tilde{p} = \tilde{p}_{j-1} = \tilde{p}_j = \text{const}$ it is easy to see that $|T_2| \le Mh^2/\max(h,\varepsilon)$.

When $p(x) \ne \text{const}$ we will expand T_2 at $\tilde{\rho}_{j-1} = p_{j-1}h/(2\varepsilon)$. Let $\tilde{r}^- = \tilde{r}^-(\tilde{\rho}_{j-1})$, $\tilde{r}^+ = \tilde{r}^+(\tilde{\rho}_j)$, $\tilde{q}^- = \tilde{q}^-(\tilde{\rho}_{j-1})$, $\tilde{q}^c = \tilde{q}^c(\rho_j)$ then $T_2 = A_1 + A_2 + A_3$, where

$$\begin{aligned}
A_1 &= h^2/2 \cdot (\tilde{r}^-(\tilde{\rho}_{j-1}) + \tilde{r}^+(\tilde{\rho}_{j-1})) + \tilde{p}_{j-1}h\tilde{q}^-(\tilde{\rho}_{j-1}) - \varepsilon(\tilde{q}^-(\tilde{\rho}_{j-1}) + \tilde{q}^c(\tilde{\rho}_{j-1})),\\
A_2 &= h^2/2 \cdot (\tilde{r}^+(\tilde{\rho}_j) - \tilde{r}^+(\tilde{\rho}_{j-1})) + h(p_{j-1} - \tilde{p}_{j-1})\tilde{q}^-(\tilde{\rho}_{j-1}),\\
A_3 &= \varepsilon(\tilde{q}^c(\tilde{\rho}_{j-1}) - \tilde{q}^c(\tilde{\rho}_j)).
\end{aligned}$$

The expression A_1 has the same form as T_2 for $p(x) = \tilde{p}_{j-1} = \text{const}$, so $|A_1| \le Mh^2/\max(h,\varepsilon)$. After some algebraic computation we get

$$|T_2W''| \le M\frac{h^2}{\max(h,\varepsilon)}(1 + \varepsilon^{-1}\exp(-2\delta x_j/\varepsilon)), \tag{3.12}$$

and that the remainder terms are bounded by

$$Mh^2(1 + \varepsilon^{-2}\exp(-2\delta\xi/\varepsilon)), \quad x_{j-1} \le \xi \le x_{j+1}. \tag{3.13}$$

From (3.12) and (3.13) we conclude that

$$|\tilde{\tau}_j(W)| \le Mh^2\varepsilon^{-1}(1 + \varepsilon^{-1}\exp(-\delta x_j/\varepsilon)) \text{ for } h \le \varepsilon. \tag{3.14}$$

Let us estimate $\tau_j(u)$. From Lemma 2.2 we have $\tilde{\tau}_j(u) = \tilde{\tau}_j(u) - \tau_{j0}(u)$, where $\tau_{j0}(u)$ denotes the truncation error for $p(x) = p(0) = \tilde{p}_{j-1} = \tilde{p}_j = \text{const}$.

$$\begin{aligned}
\tilde{\tau}_j(u) = &(T_2 - T_{20})u_j'' - (\tilde{r}^- - r_0^-)R_2(x_{j-1}, x_j, u)\\
&+ (r^+ - \tilde{r}_0^+)R_2(x_j, x_{j+1}, u) + \varepsilon(\tilde{q}^- - q_0^-)R_0(x_{j-1}, x_j, u'')\\
&- (p_{j-1}q^- - p(0)\tilde{q}_0^-)R_1(x_{j-1}, x_j, u')
\end{aligned}$$

Computing some Taylor expansions we get the estimate

$$\begin{aligned}
|T_2 - T_{20}|u_j'' &\le M(h^4x_j/\varepsilon^3 + h^2)\varepsilon^{-2}\exp(-p(0)x_j/\varepsilon)\\
&\le Mh^2\varepsilon^{-2}\exp(-\delta x_j/\varepsilon) \text{ for } h \le \varepsilon.
\end{aligned}$$

The remainder terms are bounded by $h^2\varepsilon^{-2}\exp(-\delta x_j/\varepsilon)$ for $h \le \varepsilon$, thus

$$|\tilde{\tau}_j(u)|Mh\varepsilon^{-2}\exp(-\delta x_j/\varepsilon) \text{ for } h \le \varepsilon. \tag{3.16}$$

For $\varepsilon \leq h$ we get $|T_2 - T_{20}|u_j'' \leq M \exp(-\delta x_j/\varepsilon)$.

Since $|\tilde{q}^- - q_0^-| \leq M(x_{j-1} + O(h))\varepsilon^{-1} \exp(-\delta h/\varepsilon)$ we have that

$$|p_{j-1}\tilde{q}^- - p(0)q_0^-|R_1(x_{j-1}, x_j, u') \leq M \exp(-\delta x_{j-1}/\varepsilon).$$

The estimates for the other remainders in (3.15) are similar. Thus

$$|\tau_j(u)| \leq M \exp(\delta x_{j-1}/\varepsilon) \text{ for } \varepsilon \leq h. \tag{3.17}$$

and the proof of Lemma 3.4 is completed.

Now we can prove the main Theorem 2.1. Using estimates from Lemma 3.2 and Lemma 3.4 by Lemma 3.2 we get that (2.17) holds.

4. Numerical results. We consider the following simple problem of type (1.1):

$$\varepsilon y'' + y' = x, \quad y(0) = y(1) = 0, \tag{4.1}$$

which has the solution

$$y(x) = (\varepsilon - 1/2)(1 - \exp(-x/\varepsilon))/(1 - \exp(-1/\varepsilon)) - \varepsilon x + x^2/2.$$

The numerical results for the problems above obtained by (2.15) are presented in [9]. In this paper we present the numerical results obtained by the following schemes:

I: Choosing

$$\tilde{p}_{j-1} = p(x_j - h/2) \equiv p_{j-1/2}, \quad \tilde{f}_{j-1} = f(x_j - h/2) \equiv f_{-1/2},$$
$$\tilde{p}_j = p(x_j + h/2) \equiv p_{j+1/2}, \quad \tilde{f}_j = f(x_j + h/2) \equiv f_{j+1/2}$$

(2.13) becomes $\tilde{r}_j^- \nu_{j-1}^{(0)} + r_j^C \nu_j^{(0)} + r_j^+ \nu_{j+1}^{(0)} = q_j^- f_{j-1/2} + q_j^+ f_{j+1/2}$ where

$$r_j^- = \frac{3\omega_{j-1/2}(\omega_{j-1/2} - 1)}{h(3\omega_{j-1/2}^2 - 1)}, \ r_j^+ = \frac{3\omega_{j+1/2}(\omega_{j+1/2} + 1)}{h(3\omega_{j+1/2}^2 - 1)}, \ r_j^C = r_j^- + r_j^+,$$
$$q_j^- = \frac{3\omega_{j-1/2} - 1}{p_{j-1/2}(3\omega_{j-1/2}^2 - 1)}, \ q_j^+ = \frac{3\omega_{j+1/2} + 1}{p_{j+1/2}(3\omega_{j+1/2}^2 - 1)}.$$

II: Choosing $\tilde{p}_{j-1} = p_{j-1}$, $\tilde{f}_{j-1} = f_{j-1}$, $\tilde{p}_j = p_{j+1}$, $\tilde{f}_j = f_{j+1}$, (2.13) becomes the scheme of the form $R\nu_j^{(0)} = Qf_j$, where

$$r_j^- = \frac{3\omega_{j-1}(\omega_{j-1} - 1)}{h(3\omega_{j-1}^2 - 1)}, \ r_j^+ = \frac{3\omega_{j+1}(\omega_{j+1} + 1)}{h(3\omega_{j+1}^2 - 1)}, \ -r_j^c = r_j^- + r_j^+,$$
$$q_j^- = \frac{3\omega_{j-1} - 1}{p_{j-1}(3\omega_{j-1}^2 - 1)}, \ q_j^+ = \frac{3\omega_{j+1} + 1}{p_{j+1}(3\omega_{j+1}^2 - 1)}, \ q_j^c = 0.$$

III: Choosing $\tilde{p}_{j-1} = p_{j-1}$, $\tilde{f}_{j-1} = f_{j-1}$, $\tilde{p}_j = p_j$, $\tilde{f}_j = f_j$, (2.13) becomes the scheme of the form $R\nu_{(0)}^j = Qf_j$, where

$$r_j^- = \frac{3\omega_{j-1}(\omega_{j-1}-1)}{h(3\omega_{j-1}^2-1)},\quad r_j^+ = \frac{3\omega(\omega_j+1)}{h(3\omega_j^2-2)},\quad -r_j^c + r_j^+,$$
$$q_j^- = \frac{3\omega_{j-1}-1}{p_{j-1}(3\omega_{j-1}^2-1)},\quad q_j^c = \frac{3\omega_j+1}{p_j(3\omega_j^2-1)},\quad q_j^+ = 0.$$

The numerical results are presented in Tables 1 – 3; table summarizes the results of applications of a particular scheme. For each scheme the mesh length $h = 1/\mathrm{I}$ was succesively halved starting with I= 16 and ending with I= 1024. The maximum error at all the mesh points $E_\infty = \max_j |y_j - \nu_j^{(0)}|$ is listed under E_∞. The numerical rate of convergence is determined as in Doolan et al. [3]: rate$\equiv (\ln Z_{K,\varepsilon} - \ln Z_{K+1,\varepsilon})/\ln 2$ where $Z_{K,\varepsilon} = \max|\nu_j^{h/2^K} - \nu_j^{h/2^{K+1}}|$, $K = 0,1,2,3,4$, and $\nu^{h/2^K}$ denotes the value of $\nu_j^{(0)}$ at the mesh point x_j for the mesh length $h/2^K$.

Table 1: Numerical results for I applied to (4.1)

K	Jε	1/2		1/4		1/8		1/16		1/32		1/64		1/128		1/256		1/512	
		E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate
	16	1.6 *E*-7		2.2 *E*-6		2.5 *E*-5		2.3 E-4		1.5 *E*-3		6.2 *E*-3		1.2 *E*-2		1.5 *E*-2		1.7 *E*-2	
0			4.00		3.99		3.97		3.87		3.55		2.78		1.71		1.06		0.95
	32	1.0 *E*-8		1.4 *E*-7		1.6 *E*-6		1.6 *E*-5		1.3 *E*-4		9.4 *E*-4		3.2 *E*-3		6.3 *E*-3		8.1 *E*-3	
1			4.00		3.99		3.99		3.96		3.86		3.55		2.78		1.73		1.08
	64	6.2 *E*-10		8.8 *E*-9		1.0 *E*-7		1.0 *E*-6		9.0 *E*-6		7.1 *E*-5		4.3 *E*-4		1.6 *E*-3		3.2 *E*-3	
2			4.00		4.00		4.00		4.00		3.98		3.87		3.55		2.77		1.73
	128	3.9 *E*-11		5.5 *E*-10		6.5 *E*-9		6.4 *E*-8		5.7 *E*-7		4.8 *E*-6		3.6 *E*-5		2.2 *E*-4		8.3 *E*-4	
3		.	4.00		4.0		4.00		4.00		3.99		3.97		3.87		3.55		2.78
	256	0.		3.5 *E*-11		4.0 *E*-10		4.0 *E*-9		3.6 *E*-8		3.0 *E*-7		2.5 *E*-6		1.8 *E*-5		1.1 *E*-4	
4			4.00		4.00		4.00		4.00		4.00		3.99		3.97		3.87		3.55
	512	0.		0.		2.5 *E*-11		2.5 *E*-10		2.2 *E*-9		1.9 *E*-8		1.6 *E*-7		1.2 *E*-6		9.5 *E*-6	
	1024	0.		0.		0.		1.6 *E*-11		1.4 *E*-10		1.2 *E*-9		1.0 *E*-8		8.1 *E*-8		6.4 *E*-8	

Table 2: Numerical results for II applied to(4.1)

K	J ε	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
		E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate
	16	1.5 E-4	5.3 E-4	1.5 E-3	3.4 E-3	5.2 E-3	3.0 E-3	2.4 E-3	6.1 E-3	7.9 E-3
0		2.00	1.98	1.95	1.24	1.24	—2.93	5.07	1.23	.96
	32	3.8 E-5	1.3 E-4	3.9 E-4	9.5 E-4	1.9 E-3	2.7 E-3	1.5 E-3	1.2 E-3	3.1 E-3
1		2.00	2.00	1.99	1.95	1.79	1.24	—2.93	5.10	1.25
	64	9.6 E-6	3.3 E-5	9.9 E-5	2.4 E-4	5.3 E-4	1.0 E-3	1.4 E-3	7.9 E-4	6.4 E-4
2		2.00	2.00	2.00	1.99	1.95	1.80	1.23	—2.94	5.10
	128	2.4 E-6	8.4 E-6	2.5 E-5	6.2 E-5	1.3 E-6	2.8 E-4	5.3 E-4	7.4 E-4	4.0 E-4
.		2.00	2.00	2.00	2.00	1.99	1.95	1.800	1.24	—2.93
	256	6.0 E-7	2.1 E-6	6.2 E-6	1.5 E-5	3.4 E-5	7.4 E-5	1.4 E-4	2.7 E-4	3.7 E-4
4		2.00	2.00	2.00	2.00	2.00	1.99	1.95	1.80	1.24
	512	1.5 E-7	5.3 E-7	1.5 E-6	3.8 E-6	8.7 E-6	1.8 E-5	3.8 E-5	7.6 E-5	1.3 E-4
	1024	3.7 E-8	1.3 E-7	3.9 E-7	9.7 E-7	2.1 E-6	4.6 E-6	9.6 E-6	1.9 E-5	3.8 E-5

Table 3: Numerical results for III applied to (4.1)

K	J ε	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
		E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate
	16	7.4 E-3	1.3 E-2	1.9 E-2	2.4 E-2	2.8 E-2	3.4 E-2	4.1 E-2	4.5 E-2	4.6 E-2
0		1.00	1.00	1.00	1.02	1.12	1.31	1.22	.99	.95
	32	3.7 E-3	7.5 E-3	9.6 E-3	1.1 E-2	1.3 E-2	1.5 E-2	1.8 E-2	2.1 E-2	2.3 E-2
1		1.00	1.00	1.00	1.00	1.02	1.12	1.31	1.24	1.02
	64	1.8 E-3	3.2 E-3	4.8 E-3	5.9 E-3	6.7 E-3	7.2 E-3	7.8 E-3	9.1 E-3	1.0 E-2
2		1.00	1.00	1.00	1.00	1.00	1.02	1.12	1.30	1.24
	128	9.2 E-4	1.6 E-3	2.4 E-3	2.9 E-3	3.3 E-3	3.5 E-3	3.7 E-3	4.0 E-3	4.6 E-3
3		1.00	1.00	1.00	1.00	1.00	1.00	1.02	1.12	1.30
	256	4.6 E-4	8.1 E-4	1.2 E-3	1.4 E-3	1.6 E-3	1.7 E-3	1.8 E-3	1.9 E-3	2.0 E-3
4		1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.02	1.12
	412	2.3 E-4	4.0 E-4	6.0 E-4	7.4 E-4	8.4 E-4	8.9 E-4	9.3 E-4	9.5 E-4	9.7 E4-
	1024	1.1 E-4	2.0 E-4	3.0 E-4	3.7 E-4	4.2 E-4	4.4 E-4	4.6 E-4	4.7 E-4	4.8 E-4

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(Received 19 10 1987)