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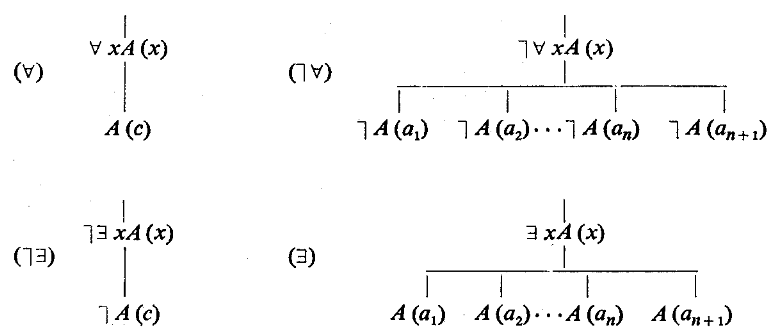
A PROOF PROCEDURE FOR THE FIRST ORDER LOGIC

Miodrag Kapetanović, Aleksandar Krapež

Abstract. A new proof procedure is given for the classical predicate logic which combines variants of the tableaux and resolution methods. Soundness and completeness of the resulting system are proved.

We shall present a proof procedure for the classical predicate logic combining features of the semantic tableaux method with the well known resolution rule of J. Robinson (see [**2**]). Given a closed first order formula (in a language without function symbols) whose validity is to be tested, we first bulid its *reduction tree* (growing downwards): put the formula at the top and then use the usual propositional tableaux rules (as described for instance in [**1**, pp. 26, 28]) as well as the following quantifier rules:



The rules apply as follows. A branch with $\forall x A(x)$ is extended by adding $A(c)$ as a new end node (the same for $(\neg\exists)$), where c is a new constant i. e. occurs in no formula on the tree. Similarly a branch containing $\exists x A(x)$ (the same for $(\neg\forall)$) gets a new extension $A(a_{n+1})$ where a_{n+1} differs from all $a_1, \dots, a_n$ (by which this occurrence of $\exists x A(x)$ has been instantiated previosly) and all $a_i (1 \leq i \leq n+1)$

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occur already on the tree. If there are no constants on the tree at all then instantiate $\exists x A(x)$ by a new one. All (occurrences of all) formulas are used at most once w. r. t. a patricular branch and discharged after that. As an exception an ocurrence of $\exists x A(x)$ or $\neg\forall x A(x)$ is discharged only in case it cannot be instantiated by any constant anymore and no new constant can appear on the tree by any further reduction. By *basic sentences* of our language, enriched by new constants, we mean atomic sentences and their negations. The reduction trees we are most interested in are finished ones i. e. those on which all ocurrences of all formulas are discharged. They have the fundamental property expressed by the following.

LEMMA. *Given a finished reduction tree T^* of a closes formula, the formula is valid iff for every model (in the language of T^*) there is a branch of T^* such that all basic sentences from that branch hold in the model.*

Proof. ($\Leftarrow$) Suppose that a given formula F is not valid i. e. there is an M such that $M \not\models F$. Define an expansion M^* of that model as follows. First enumerate all constants appearing in T^* into a sequence $c_0, c_1, \ldots$ and let $M_0 = M$. Then for all $n < \omega$ let the language $L(M_{n+1}) = L(M_n) \cup \{c_n\}$. Suppose that c_n was introduced by applying ($\forall$) to $\forall x A(x)$ (the same in case of ($\neg\exists$)). Then if $M_n \not\models \forall x A(x)$ take some $e \in \mid M \mid$ such that $M_n \not\models A[e]$ and put $c_n^{M_{n+1}} = e$. Otherwise $c_n^{M_{n+1}}$ is arbitrary. Finally let $\mid M^* \mid = \mid M \mid$, $L^* = L(M^*) = \bigcup_{n<\omega} L(M_n)$, $c_n^{M^*} = c_n^{M_{n+1}}$. Obviously $M^* \not\models F$ but suppose there is a branch $\mathcal{B}$ of T^* whose all basic sentences hold in M^*. Then by induction it follows that all sentences from $\mathcal{B}$ (including F) hold in M^*: if $A \wedge B \in \mathcal{B}$ then both $A, B \in \mathcal{B}(\mathcal{B}$ is finished!) and they hold in M^* by induction hypothesis (the same argument for $\neg\neg A, \neg(A \vee B), \neg(A \Rightarrow B)$). If $A \vee B \in \mathcal{B}$ then $A \in \mathcal{B}$ or $B \in \mathcal{B}$, say $A \in \mathcal{B}$, and again $M^* \models A$ by induction hypothesis (the same for $\neg(A \wedge B), A \Rightarrow B$). If $\forall x A(x) \in \mathcal{B}$ (the same for $\neg\exists x A(x)$) then $A(c_n) \in \mathcal{B}$ for some n and $M^* \models A(c_n)$ by induction hypothesis. Using the definition of $c_n^{M^*}$ we get $M^* \models \forall x A(x)$. Finally if $\exists x B(x) \in \mathcal{B}$ then again $B(c_n) \in \mathcal{B}$ for some n and $M^* \models B(c_n)$ by induction hypothesis. As $F \in \mathcal{B}$ we get the contradiction.

($\Rightarrow$) Suppose that F is valid and let M be any structure in the language L^*. Take the restriction $M^C = M \uparrow \{e \in \mid M \mid\mid$ for some constant $c, c^M = e\}$ and choose a branch $\mathcal{B}$ by defining inductively its initial segments $\mathcal{B}_n = (F_0, F_1, \ldots, F_n)$, $n = 0, 1, \ldots$ (we identify a node of T^* with the formula placed at it): if F_n has only one successor take it for F_{n+1}. If it has at least two successors choose one (say the leftmost) that holds in M^C. We shall show by induction on n that $M^C \models F_n$ and the same time see that our definition is correct. Certainly $M^C \models F_0$, so suppose F_{n+1} is the only successor of F_n. It could only appear as a component of some $F_i (i \leq n)$ of the form $\neg\neg A$ or $A \wedge B$ or $\neg(A \Rightarrow B)$ or $\neg(A \vee B)$ or $\forall x A(x)$ or $\neg\exists x A(x)$. Take for example the case $F_i = A \wedge B$, $F_{n+1} = A$. Then $M^C \models F_i$ by induction hypothesis so $M^C \models F_{n+1}$. If F_i is some $\forall x A(x)$ then for some c F_{n+1} is $A(c)$, so obviously $M^C \models F_{n+1}$ e. t. c. Suppose now that F_{n+1} has been chosen among two or more successors of F_n. Then it must be a component of some F_i of the form $A \vee B$ or $A \Rightarrow B$ or $\neg(A \wedge B)$ or $\exists x A(x)$, or $\neg\forall x A(x)$. If for instance F_i is some $A \vee B$ and F_{n+1} is B then $M^C \models F_i$ by induction hypothesis, so $M^C \models A$ or $M^C \models B$. The choice of B implies that $M^C \not\models A$, hence M^C must satisfy B whish shows both the possibility and correctness of our choice. The most interesting case is that of $\exists x A(x)$ (and $\neg\forall x A(x)$). If $M^C \models \exists x A(x)$ then $M^C \models A[e]$ for some $e \in \mid M^C \mid$. But $e = c^{M^C}$ for some c by the definition of M^C, so again there exists a successor of F_n true in M^C proving that F_{n+1} has been well defined. Finally, this shows that the whole of $\mathcal{B}$ holds in M^C (including all basic sentences of $\mathcal{B}$) i. e. the lemma is proved, since M was chosen arbitrarily.

To get a complete set of rules we have to add the following *dual resolution rule* to the reduction rules given already:

(DR) Let S_1 and S_2 be sets of basic sentences, $A \in S_1$ and $\neg A \in S_2$; then we *resolve* S_1 and S_2 to get their resolvent $R(S_1,\ S_2,\ A) =$ $= (S_1 \backslash \{A\} \cup (S_2 \backslash \{\neg A\})$.

In derivations we allow only those S_i which are either sets of all basic sentences from a finished branch or resolvents obtained therefrom. A formula is proved if we can derive the empty set from it, using (DR) and the reduction rules.

As is well known, the existence of a model of a set of basic sentences is equaivalent to the existence of the apropriate valuations where we take these sentences as propositional variables. If a valuation satisfies at least one of the sets of basic sentences from $R(S_1, S_2, A), S_3, \dots, S_k$ then it satisfies at least one from $S_1, S_2, S_3, \dots, S_k$. The proof sistems from the tautology $(B \wedge C) \Rightarrow ((B \wedge A) \vee (C \wedge \neg A))$ since we can look at S_i's as conjuctions of their members.

Since every valuation satisfies the empty set, it follows that every valuation satisfies at least one among the sets of basic sentences from the finished branches from which we derived the empty set. Together with $(\Rightarrow)$ of Lemma this implies that our formula is valid, i. e. we can prove only valid formulas.

To show the converse take a valid formula and consider its finished reduction tree T^*. Then the set $\{\Gamma \mid \Gamma$ is the set of all basic sentences of some bransh of $T^*\}$ has, by $(\Leftarrow)$ of Lemma, the property that for any valuation at least one of its members is true. By the compactness theorem there is a finite subset $\{\Gamma_1, \dots, \Gamma_m\}$ with the same property. Using ideas from [**3**] and [**2**] we shall prove (by induction on $n = \mid \Gamma_1 \mid + \cdots + \mid \Gamma_m \mid -m$) that we can infer $\varnothing$ from this subset ($\mid \Gamma_i \mid$ is cardinality of Γ_i).

If $n = 0$ then all Γ_i contain only one element so there are $j, k \leq m$ and an atomic sentence A such that $\Gamma_j\{A\},\ \Gamma_k = \{\neg A\}$. So the basis of induction is true.

For $n > 0$ take Γ_i with at least two elements and represent it as a disjoint union of nonempty sets Δ_1 and Δ_2. By the induction hypothesis $\varnothing$ can be inferred from $\Gamma_1, \dots,\ \Gamma_{i-1},\ \Delta_1,\ \Gamma_{i+1}, \dots, \Gamma_m$. Repeating the same steps in this inference, with Δ_1 replaced by $\Delta_1 \cup \Delta_2 (= \Gamma_i)$ we get the inference of Δ_2 from $\Gamma_1, \dots, \Gamma_m$. Using induction hypothesis again we can extend this inference to an inference of $\varnothing$ from $\Gamma_1, \dots,\ \Gamma_{i-1},\ \Delta_2,\ \Gamma_{i+1}, \dots,\ \Gamma_m$ thus obtaining the inference of $\varnothing$ from $\Gamma_1, \dots \Gamma_m$.

This proves the *completeness* of our system: only valid formulas are provable.

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Matematički Institut
Knez Mihailova 35,
11000 Beograd, P.P. 367
Yugoslavia

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