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PERFECT MATCHINGS IN A CLASS OF BIPARTITE GRAPHS

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Abstract. Some relations for the number of perfect matchings in a class of graphs are established.

In this paper we consider undirected graphs without loops and multiple edges. Let $I_p = \{i_1, i_2, \dots, i_{2p}\} \subset \{1, 2, \dots, n\}$ and $i_j < i_{j+1}$, $j = 1, \dots, 2p-1$. Consider a graph $G(n, I_p)$ having n vertices. These vertices are labeled by $1, 2, \dots, n$ and the following edges exist in $G(n, I_p)$: $(i, i+1), i = 1, 2, \dots, n-1$; $(1, n)$; (i_j, i_{2p-j+1}), $j = 1, \dots, p$. It is further required that $i_{2p} - i_1 < n-1$ and $i_{p+1} - i_p > 1$, otherwise we would have to allow multiple edges in $G(n, I_p)$.

The structure of $G(n, I_p)$ is presented in Fig. 1. From Fig. 1 it is easy to conclude that $G(n, I_p)$ will be bipartite if n is even and $i_{2p-j+1} - i_j \equiv 1 \pmod 2$ for $j = 1, \dots, p$.

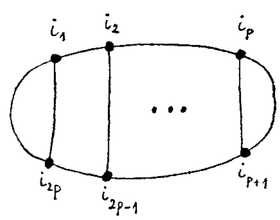

Fig. 1

If G is a graph possessing n vertices and n is even, then a perfect matching $M(G)$ of G is a set of $n/2$ edges of G, such that if $(u, v) \in M(G)$ and $(w, z) \in M(G)$, then $| \{u, v, w, z\} | \neq 3$.

The number of distinct perfect matchings of the graph G is denoted by $k(G)$.

In this paper we establish several results for $k(G(n, I_p))$ when $G(n, I_p)$ is bipartite. In the discussion which follows is always assumed that $G(n, I_p)$ is bipartite.

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THEOREM 1. *If* $p = 1$, *then* $k(G(n, I_p)) = 3$. *If* $p = 2$, *then* $k(G(n, I_p)) = [9 + (-1)^{i_2 - i_1}]/2$. *If* $p > 2$, *then* $k(G(n, I_p))$ *is uniquely determined by the ordered sequence* $S = [S_1, S_2, \dots, S_{p-1}]$ *of symbols* E *(even) and* O *(odd), defined as*

$$S_j = \begin{cases} E & \textit{if} \quad i_{j+1} - i_j \equiv 0 \ (mod\ 2) \\ O & \textit{if} \quad i_{j+1} - i_j \equiv 1 \ (mod\ 2). \end{cases}$$

In order to prove Theorem 1 we need an auxiliary result.

Let G be a graph and v_1, v_2, v_3, v_4 its distinct vertices, such that v_1 and v_{i+1} are adjacent, $i = 1, 2, 3, v_1$ and v_4 are not adjacent, and v_2 and v_3 have degree two. Let the graph H be obtained by deleting from G the vertices v_2 and v_3 and by joining v_1 and v_4.

LEMMA 1. $k(H) = k(G)$.

Proof. We demonstrate a one-to-one correspodence between the perfect matchings of G and H.

Let $M'(G)$ be a perfect matching of G containing the edge (v_1, v_2). Then necessarily $(v_2, v_3) \notin M'(G)$, $(v_3, v_4) \in M'(G)$. The corresponding perfect matching of H is $M'(H) = M'(G) \backslash \{(v_1, v_2), (v_3, v_4)\} \cup \{(v_1, v_4)\}$. Note that (v_1, v_4) belongs to $M'(H)$.

Let $M''(G)$ be a perfect matching of G not containing (v_1, v_2). Then $(v_2, v_3) \in M''(G)$, $(v_3, v_4) \notin M''(G)$. The corresponding perfect matching of H is $M''(H) = M''(G) \backslash \{(v_2, v_3)\}$. Note that $(v_1, v_4) \notin M''(H)$.

Since any perfect matching of G is either of type $M'(G)$ or $M''(G)$, and any perfect matching of H is either of type $M'(H)$ or $M''(H)$, the correspodence described above is a bijection. □

Proof of Theorem 1. For $p = 1$ and $p = 2$ the statement of Theorem 1 can be easily verified by direct checking. Therefore we focus our attention on the case $p > 2$.

Denote by $q = q(S)$ the number of times the symbol E occurs in the sequence S.

As an immediate consequence of Lemma 1, whenever for some $j = 1, \dots, p-1$, $p+1, \dots 2p-1$ we have $i_{j+1} - i_j \geq 3$, we can perform a "contraction" of $G(n, I_p)$ by reducing by two the number of vertices laying between i_j and i_{j+1}; this transformation does not affect the value of k. Similar contractions can be performed between i_p and i_{p+1} provided $i_{p+1} - i_p > 3$, and between i_1 and i_n provided $i_1 + n - i_n > 3$.

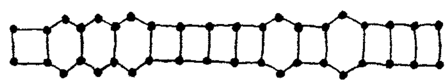
Applying the contraction as many times as possible, we finally arrive at the graph $G(n^*, I_p^*)$ for which $n^* = 4p - 2q + 2$, $I_p^* = \{i_1^*, i_2^*, \ldots, i_{2p}^*\}$ and

$$
\begin{aligned}
& i_1^* = 2 \\
& i_{j+1}^* - i_j^* = i_{2p-j+1}^* - i_{2p-j}^* = \begin{cases} 1 & \text{if } S_j = E \\ 2, & \text{if } S_j = O \end{cases} \quad j = 1, 2, \ldots, p-1 \\
& i_{p+1}^* - i_p^* = 3 \\
& i_{2p}^* = n^* - 1.
\end{aligned}
$$

The contracted graph $G(n^*, I_p^*)$ has the same number of perfect matchings as $G(n, I_p)$. On the other hand, it is clear that the structure of the graph $G(n^*, I_p^*)$ is fully determined by the sequence $\mathbf{S}$. $\square$

Bearing in mind Theorem 1, we shall denote the number of perfect matchings of $G(n, I_p)$ by $k(\mathbf{S})$. The contracted graph corresponding to $\mathbf{S}$ will be denoted by $G(\mathbf{S})$.

A typical graph of the type $G(\mathbf{S})$ is depicted in Fig. 2. Such graphs consist of a linear array of squares and hexagons. The number of squares and hexagons is $q + 2$ and $p - q - 1$, respectively.



$G(S)$; $S = [OOOEEEEOEOEE]$

Fig. 2

THEOREM 2. *For $i = 1, \ldots, p-1$ define the matrices $\mathbf{X}_i$ as*

$$
\mathbf{X}_i = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ if } S_i = E; \qquad \mathbf{X}_i = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \text{ if } S_i = 0.
$$

Then $k(\mathbf{S}) = 3(\mathbf{X}_1 \mathbf{X}_2 \ldots \mathbf{X}_{p-1})_{11} + 2(\mathbf{X}_1 \mathbf{X}_2 \ldots \mathbf{X}_{p-1})_{12}$.

Theorem 2 is equivalent to a result proved in [**1**]. We mention it for completeness, and because of its formal similarity with Theorem 3.

For $p > 2$ the sequence S can be presented as

$$
\mathbf{S} = [O^{t_0} E O^{t_1} E O^{t_2} \ldots E O^{t_p}] \tag{1}
$$

where $t_i \geq 0$, and where use the convention $OO = O^2$, $OOO = O^3$, $OOOO = O^4, \ldots$, and also $EO^0E = EE$.

THEOREM 3. *Let the sequence $\mathbf{S}$ be of the form (1). For $i = 0, 1, \ldots, q$ define the matrices $\mathbf{Y}_i$ as*

$$
\mathbf{Y}_i = \begin{pmatrix} t_i + 1 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Then $k(\mathbf{S}) = (\mathbf{Y}_0 \mathbf{Y}_1, \ldots \mathbf{Y}_q)_{11} + (\mathbf{Y}_0 \mathbf{Y}_1 \ldots \mathbf{Y}_q)_{12} + (\mathbf{Y}_0 \mathbf{Y}_1 \ldots \mathbf{Y}_q)_{21} + (\mathbf{Y}_0 \mathbf{Y}_1 \ldots \mathbf{Y}_q)_{22}$.

Denote the edges $(1, n^*)$ and $(i_p^* + 1, i_p^* + 2)$ of the graph $G(\mathbf{S})$ by $e_1 = e_1(\mathbf{S})$ and $e_2 = e_2(\mathbf{S})$, respectively. Let further $k_{11}(\mathbf{S})$, $k_{12}(\mathbf{S})$, $k_{21}(\mathbf{S})$ and $k_{22}(\mathbf{S})$ denote the number of perfect matchings of $G(S)$, which contain respectively e_1 and e_2, only e_1, only e_2, and neither e_1 nor e_2. Then

$$k(\mathbf{S}) = k_{11}(\mathbf{S}) + k_{12}(\mathbf{S}) + k_{21}(\mathbf{S}) + k_{22}(\mathbf{S}). \tag{2}$$

In order to deduce Theorem 3 we prove a somewhat stronger result. Denote the matrix product $\mathbf{Y}_0\mathbf{Y}_1 \ldots \mathbf{Y}_q$ by $\mathbf{Y}(\mathbf{S})$.

LEMMA 2.

$$\mathbf{Y}(\mathbf{S})_{ij} = k_{ij}(\mathbf{S}), \qquad i, j \in \{1, 2\} \tag{3}$$

It is evident that Theorem 3 is an immediate corollary of Lemma 2 and eq. (2).

Proof of Lemma 2. We make an iduction on q, the number of symbols E in $\mathbf{S}$.

First, if $q = 0$, then eq. (3) is easily verified.

Consider now two sequences $\mathbf{S}'$ and $\mathbf{S}''$ of symbols E and O. Denote by $\mathbf{S}' \oplus \mathbf{S}''$ the sequence in which the elements of $\mathbf{S}'$ are followed by a symbol E and then by the elements of $\mathbf{S}''$. Suppose that eq. (3) holds for $\max\{q(\mathbf{S}'), q(\mathbf{S}'')\}$. Then

$$\mathbf{Y}(\mathbf{S}' \oplus \mathbf{S}'') = \mathbf{Y}(\mathbf{S}')\mathbf{Y}(\mathbf{S}''). \tag{4}$$

In order to obtain the identity (4) we analyse the perfect matchings of $G(\mathbf{S}' \oplus \mathbf{S}'')$. The newly added symbol E in $\mathbf{S}' \oplus \mathbf{S}''$ corresponds to a square in the graph $G(\mathbf{S}' \oplus \mathbf{S}'')$. Two of the four edges of this square lie on the boundary of $G(\mathbf{S}' \oplus \mathbf{S}'')$; they are denoted by f_1 and f_2. The two additional edges, which do not belong to the boundary, are denoted by f_3 and f_4; see Fig. 3.

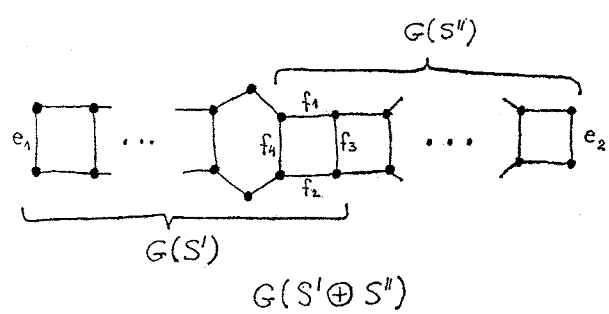


Fig. 3

Since we have resticed our consideration to bipartite graphs, it is not difficult to see that a perfect matching of $G(\mathbf{S}' \oplus \mathbf{S}'')$ either contains both f_1 and f_2 or none of them.

We first examine those perfect matchings of $G(\mathbf{S}' \oplus \mathbf{S}'')$ which contain both of the edges e_1 and e_2 (see Fig. 3). Their number is $k_{11}(\mathbf{S}' \oplus \mathbf{S}'')$. Among these perfect matchings some contain f_1 and f_2, and some not.

Perfect matchings which contain f_1 and f_2 cannot contain f_3 and f_4. Observing that $f_3 = e_2(\mathbf{S}')$ and $f_4 = e_1(\mathbf{S}'')$, we conclude that the number of such perfect matchings is $k_{11}(\mathbf{S}')k_{11}(\mathbf{S}'')$.

For the same reason the number of perfect matchings which contain e_1 and e_2, but not f_1 and f_2, is equal to $k_{12}(\mathbf{S}')k_{21}(\mathbf{S}'')$.
This gives

$$k_{11}(\mathbf{S}' \oplus \mathbf{S}'') = k_{11}(\mathbf{S}')k_{11}(\mathbf{S}'') + k_{12}(\mathbf{S}')k_{21}(\mathbf{S}'')$$

or, by taking into account the induction hypothesis,

$$k_{11}(\mathbf{S}' \oplus \mathbf{S}'') = \mathbf{Y}(\mathbf{S}')_{11}\mathbf{Y}(\mathbf{S}'')_{11} + \mathbf{Y}(\mathbf{S}')_{12}\mathbf{Y}(\mathbf{S}'')_{21}.$$

This means that the relation

$$k_{ij}(\mathbf{S}' \oplus \mathbf{S}'') = [\mathbf{Y}(\mathbf{S}')\mathbf{Y}(\mathbf{S}'')]_{ij} \tag{5}$$

is valid for $i = j = 1$.

The remaining three relations of type (5) are deduced by using a completely analogous reasoning. Hence (5) holds for $i, j \in \{1, 2\}$.

If we choose the sequence $\mathbf{S}''$ so that $q(\mathbf{S}'') = 0$, then $q(\mathbf{S}' \oplus \mathbf{S}'') = q(\mathbf{S}') + 1$. Therefore (5) implies that if (3) holds for sequences $\mathbf{S}$ possessing q symbols E, then it will also hold for sequences possessing $q + 1$ symbols E.

This proves Lemma 2 and therefore also Theorem 3. □

COROLLARY 3.1. *The numbers $k_{ij}(\mathbf{S})$ obey the identity*

$$k_{11}(\mathbf{S})k_{22}(\mathbf{S}) - k_{12}(\mathbf{S})k_{21}(\mathbf{S}) = (-1)^{p+1}.$$

Proof. Corollary 3.1. is just another way to state that det $\mathbf{Y}(\mathbf{S}) = (-1)^{p+1}$. This latter relation follows from $\mathbf{Y}(\mathbf{S}) = \mathbf{Y}_0\mathbf{Y}_1 \dots \mathbf{Y}_p$ and the obvious fact that det $\mathbf{Y}_i = -1,\ i = 0, 1, \dots, q$.□

COROLLARY 3.2. *Cyclic permutations of the factors do not alter the trace of the product $\mathbf{Y} = \mathbf{Y}_0\mathbf{Y}_1\mathbf{Y}_2, \dots \mathbf{Y}_q$.*

Proof. It is sufficient to demonstrate that the above statement is true for $\mathbf{Y}' = \mathbf{Y}_1\mathbf{Y}_2 \dots \mathbf{Y}_n\mathbf{Y}_0$. Let $t_0 + 1 = a$. Then

$$\mathbf{Y}' = \mathbf{Y}_0^{-1}\mathbf{Y}\mathbf{Y}_0 = \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix} \mathbf{Y} \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

and therefore, $Y'_{11} = aY_{21} + Y_{22},\ Y'_{22} = Y_{11} - aY_{21}$. Hence, $Y'_{11} + Y'_{22} = Y_{11} + Y_{22}$.□

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