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ON THE LARGEST EIGENVALUE OF BICYCLIC GRAPHS

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Abstract. Among bicyclic graphs (connected graphs with two independent cycles) we find those graphs whose largest eigenvalue (index, for short) is minimal.

1. Introduction

We will consider only finite, undirected graphs, without loops or miltiple lines. Our basic terminology follows [**6**]; for everything about graph spectra, not given here, see [**4**].

There is an increasing interest in literature for examining the largest eigenvalue or the index of a graph (according to [**4**]). For many families of graphs, the problem of finding all those graphs whose index is either minimal, or maximal, has already, been solved. In particular, see [**4**,**8**] for trees, or [**3**,**4**,**12**] for unicyclic graphs. More generally, for connected, or even disconnected graphs, with prescribed number of points and lines, the problem of finding those graphs whose index is maximal has been widely studied in literature (see [**2**,**3**,**5**,**9**], for example). On the other hand, there are not too many results related to the minimal index problem. Here we will provide a solution of the latter problem for bicyclic graphs, i.e., connected graphs with two independent cycles. Acctually, we will prove some facts already announced in [**10**]. For some accounts on tricyclic graphs see [**1**].

2. Bicyclic graphs with the minimal index

Consider the set of all bicyclic graphs on a prescribed number of points. Our aim is to find a graph (or graphs) from this set whose index is minimal. As remarked in [**11**] no such graph have a point of degree one; otherwise, we can remove that point and reinsert it in any line belonging to some cycle (see also [**7**]). Thus the candidates for the graphs we are looking for are of the types as depicted in Fig. 1.

According to [**11**], we in further have:

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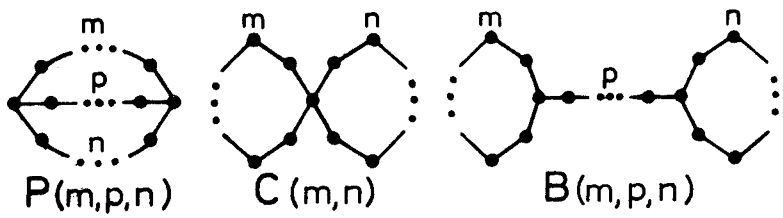

Fig. 1*

(i) for the graphs of the first two types the index decreases monotonously as the corresponding parameters become closer to each other, i.e. it is the smalest if the related parameters are as equal as possible ($\max(m,p,n)-\min(m,p,n) \leq 1$ in the former case, or $\max(m,n)-\min(m,n) \leq 1$ in the latter case);

(ii) the index of the (unique) candidate of the first type is strictly less than the index of the corresponding candidate of the second type.

So to end up with all graphs in question we have to examine the graphs of the third type. In that respect, it is worthwhile to mention (see also [**11**]) that, if $m = n$ in some graph of the third type, then its index is the same as the index of the graph of the first type with the same parameters. Thus the graphs of the third type with $m \neq n$ have to be examined.

Let $G = B(m,p,n)$ (see Fig. 1) be any graph of the third type, and $\rho = \rho(G) = \rho(m,p,n)$ its index. For convencience let

$$x_0, x_1, \ldots, x_{m-1}, z_1, \ldots, z_{p-1}, y_0, y_1, \ldots, y_{n-1}$$

(with $x_0 = x_m = z_0 = a$ and $y_0 = y_n = z_p = b$) be the coordinates of the eigenvector corresponding to the index. Thus, the condition

$$\rho\xi(v) = \sum_{u \sim v} \xi(u) \tag{1}$$

must hold for any point v of G.

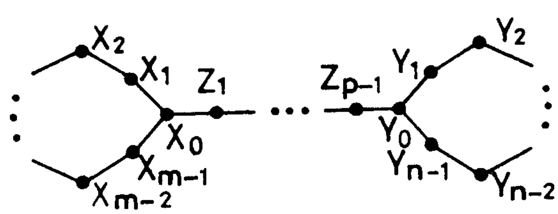

Fig. 2

* m, n, p denote path or cycle lengths.

The following lemma can be easily verified.

LEMMA 1. *The difference equation*

$$c_{i+2} - rc_{i+1} + c_i = 0 \quad (i = 0, \dots, k-2); \qquad c_0 = a,\ c_k = b \tag{2}$$

has the following solution

$$c_i = F_i(t, k, a, b) = (b \operatorname{sh} it + a \operatorname{sh}(k-i)t) / \operatorname{sh} kt,$$

where $t = \ln\bigl(r + \sqrt{r^2 - 4}\bigr)/2$ *(or* $r = 2 \operatorname{ch} t$*).*

Now we first remark that if $r = \rho$ and $\deg v = 2$ then (1) is reduced to (2) for an appropriate choice of k and initial conditions. So ρ is actually an eigenvalue of G if, in addition, (1) holds for both points of degree three. Since $\rho = 2 \operatorname{ch} t$ (by Lemma 1), this requirement gives

$$\begin{aligned} 2a \operatorname{ch} t &= 2F_1(t, m, a, a) + F_1(t, p, a, b) \\ 2b \operatorname{ch} t &= 2F_1(t, n, b, b) + F_{p-1}(t, p, a, b), \end{aligned} \tag{3}$$

where we have already used the fact that $F_{m-1}(t, m, a, a) = F_1(t, m, a, a)$ and $F_{n-1}(t, n, b, b) = F_1(t, n, b, b)$. Putting

$$f(t, k) = F_1(t, k, 1, 1) = (\operatorname{sh} t + \operatorname{sh}(k-1)t) / \operatorname{sh} kt \tag{4}$$

from (3) we get

$$\begin{aligned} f(t, m) + \frac{1}{2} f(t, p) - \operatorname{ch} t &= \frac{a-b}{2a} \frac{\operatorname{sh} t}{\operatorname{sh} pt}, \\ f(t, n) + \frac{1}{2} f(t, p) - \operatorname{ch} t &= \frac{b-a}{2b} \frac{\operatorname{sh} t}{\operatorname{sh} pt}. \end{aligned} \tag{5}$$

Eliminating a and b from (5), and also putting

$$g(t, k) = f(t, k) + (1/2) f(t, p) - \operatorname{ch} t \tag{6}$$

we get our basic relation

$$\frac{1}{g(t, m)} + \frac{1}{g(t, n)} = 2 \frac{\operatorname{sh} pt}{\operatorname{sh} t}. \tag{7}$$

For convenience, we rewrite it in the form

$$G(t, m, n) = H(t, p), \tag{7'}$$

where

$$G(t, m, n) = \frac{1}{g(t, m)} + \frac{1}{g(t, n)}, \tag{8}$$

$$H(t, p) = 2 \frac{\operatorname{sh} pt}{\operatorname{sh} t}. \tag{9}$$

To examine the behaviour of $\rho(m,p,n)$, we will first keep p fixed. The following four lemmas, which immediately follow from the elementary calculus, will be used in the sequel.

Lemma 2. *For $p>1$ and $t>0$, $\operatorname{sh} pt/\operatorname{sh} t$ is increasing in t.*

Lemma 3. *For $k=1$, $f(t,k)=1$; for $k>1$ and $t>0$, $f(t,k)$ is decreasing in t. Moreover, if $j>i$, then $f(t,j)<f(t,i)$.*

Remark 1. From (5) and the above lemma, it immediately follows that $a>b$, whenever $m<n$.

Lemma 4. *For $t>0$, $g(t,k)$ is decreasing in t and tends to $-\infty$.*

Denote by t_2 (t_1) the unique zero of $g(t,m)$ (resp. $g(t,n)$); provided $m<n$, according to Lemma 3, $t_1<t_2$. From (5), since a and b are both positive (or negative), it follows that $g(t,m)$ and $g(t,n)$ are of different signs near a point $t=t_\rho$ which is a solution in t of the equation (7) that corresponds to the index of the observed graph. More precisely, we can take that $t_\rho>t_0$, where t_0 is a point where $G(t,m,n)$ vanishes.

Lemma 5. *If $t_0<t<t_2$, then $G(t,m,n)$ is strictly increasing.*

Lemma 6. *If $m+n$ is fixed and $m<n$, then $G(t,m,n)$ is increasing in m, for any t belonging to (t_0,t_2).*

Proof. Deriving $G(t,m,n)$ with respect to m we get

$$\begin{aligned}\frac{\partial}{\partial m}G(t,m,n)&=-g(t,m)^{-2}\frac{\partial}{\partial m}g(t,m)-g(t,n)^{-2}\frac{\partial}{\partial m}g(t,n)\\&=\frac{t}{2}\operatorname{sh} t\,\frac{g^2(t,n)\operatorname{ch}^2(nt/2)-g^2(t,m)\operatorname{ch}^2(mt/2)}{g^2(t,m)g^2(t,n)\operatorname{ch}^2(mt/2)\operatorname{ch}^2(nt/2)}.\end{aligned}$$

The latter expression is positive since

(i) $\operatorname{ch}(nt/2)>\operatorname{ch}(mt/2)$; (ii) $|g(t,n)|>|g(t,m)|$ if $t_0<t<t_2$. $\square$

The following lemma is also useful. It is a direct consequence of the Lagrange's theorem on finite differences.

Lemma 7. *Let $f,g:(a,b)\to\mathbf{R}$ be differentiable functions satisfying*: (i) $\lim_{x\to a}=0$, $\lim_{x\to b}f(x)=+\infty$; (ii) *$g(x)$ is bounded on (a,b)*; (iii) *there exists a unique point c $(a<c<b)$ such that $f(c)=g(c)$. Then $f'(c)>g'(c)$.*

Proposition 1. *Let $\rho(m,p,n)$ be the index of G. If p and $m+n$ are fixed, then provided $m<n$, $\rho(m,p,n)$ is decreasing in m.*

Proof. Since $\rho=2\operatorname{ch} t$, it is sufficient to show that the partial derivative of t $(=t(m,p,n))$ with respect to m is negative. From (7) we get

$$\frac{\partial}{\partial m}t=-\frac{\partial G/\partial m}{\partial H/\partial t-\partial G/\partial t}.$$

By Lemma 6, $\partial G/\partial m$ is positive if $m < n$. Applying Lemma 7, we also get that $\partial H/\partial t - \partial G/\partial t$ is positive at the point $t = t_\rho$. $\square$

Corollary 1. *If $m+n$ is even $(=2k)$, then $\rho(k,p,k) \le \rho(m,p,n)$ with equality when $k = (m+n)/2$; otherwise, if $m+n$ is odd $(=2k+1)$ and $p \ne k$, then*

$$\rho(k,p,k+1) > \begin{cases} \rho(k,p+1,k) & \text{if } p < k \\ \rho(k+1,p-1,k+1) & \text{if } p > k. \end{cases} \tag{10}$$

Note that the latter part of the Corollary follows from Proposition 1 by making use of $\rho(k,p,k+1) > \rho(k+1/2,p,k+1/2)$ and a result from [**11**] which asserts that $\rho(x,y,z)$, provided $x+y+z$ is fixed, is minimum if x, y and z are as equal as possible.

Finally, we have to prove that

$$\rho(k,k,k+1) > \rho(k,k+1,k) \tag{11}$$

which cannot be done by the arguments used above.

Let $\sigma = \sigma(k,k,k+1)$. Also, let $t_\sigma = \ln\bigl(\sigma + \sqrt{\sigma^2-4}\bigr)/2$. Consider then a vector ξ' constructed as follows

$$\begin{aligned} x_i' = z_i' &= F_i(t_0,k,a,a) \qquad (i=0,\dots,k-1) \\ y_i' &= F_i(t_0,k+1,a,a) \qquad (i=0,\dots,k) \end{aligned}$$

which corresponds to the graph $G = B(k,k+1,k)$, in accordance with Fig. 2. By Lemma 1, (1) holds with each point of degree two no matter of the choice of σ. Observing any point of degree three (denote it by v), we will show that the expression defined by

$$\varphi(v) := \sigma\xi'(v) - \sum_{u\sim v} \xi'(u) \tag{12}$$

is positive. Namely, due to (5) we have

$$\begin{aligned} \varphi(v) &= 2a\operatorname{ch} t_0 - 2aF_1(t_0,k,1,1) - aF_{k+1}(t_0,k+1,1,1) \\ &= 2a\operatorname{ch} t_0 - 2af(t_0,k) - af(t_0,k+1) \\ &= (a-b)^2 b^{-1} \operatorname{sh} t_0/\operatorname{sh} pt_0 > 0. \end{aligned}$$

So, if A is the adjacency matrix of G, we have found a positive vector ξ', and a scalar σ, such that $A\xi' \le \sigma\xi'$ and $A\xi' \ne \sigma\xi'$. Thus, the index of G is strictly less than σ, i.e. (11) holds.

Now we are in position to state our main result.

Theorem 1. *Let $\mathcal{B}_n$ $(n \ge 7)$ be the set of all bicyclic graphs on n points, and let $k = \lceil n/3 \rceil$. There are two graphs in $\mathcal{B}_n$ whose index is minimal; one of them is $P(k,n+1-2k,k)$, while the other is $B(k,n+1-2k,k)$ (see Fig. 1.).*

Remark 2. According to [**3**], there is a unique graph in the above set whose index is maximal. It is the star with two lines having a common point being added.

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