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VERTEX DEGREE SEQUENCES OF GRAPHS WITH SMALL NUMBER OF CIRCUITS

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Abstract. Necessary and sufficient conditions are determined for the numbers $p_1, p_2, \dots, p_n$ to be the vertex degrees of a connected graph with n vertices and cyclomatic number c, $c = 0, 1, 2, 3, 4, 5$.

Introduction

A partition $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ of the number $2m$ is said to be graphic if there exists a graph G with n vertices and m edges, such that the degree of the i-th vertex of G is equal to p_i, $i = 1, 2, \dots, n$. The characterization of graphic partitions and the study of graphs with prescribed degree sequences is a well elaborated part of graph theory [**1**,**2**].

Denote by $\mathbf{P}_N$ the set of all partitions of the integer N. If $\boldsymbol{a} \in \mathbf{P}_N$ then we say that $\boldsymbol{a}$ is of order N. Further, we present $\boldsymbol{a}$ as $(a_1, a_2, \dots, a_\alpha)$ and assume that $a_1 \geq a_2 \geq \cdots \geq a_\alpha > 0$. Of course, $a_1 + a_2 + \cdots + a_\alpha = N$.

If $\boldsymbol{a} \in \mathbf{P}_N$ then the conjugate partition of $\boldsymbol{a}$ is denoted by $\boldsymbol{a}^*$ and is defined as $\boldsymbol{a}^* = (a_1^*, a_2^*, \dots, a_{\alpha^*}^*)$ where $\alpha^* = a_1$ and $a_j^* = \max\{i \mid a_i \geq j\}$, $j = 1, 2, \dots, \alpha^*$. Then $\boldsymbol{a}^* \in \mathbf{P}_N$ and $a_1^* \geq a_2^* \geq \cdots \geq a_{\alpha^*}^* > 0$.

The partition $\boldsymbol{a}$ can be visualized by means of a Ferrers diagram [**1**] which is obtained by setting a_i dots in the i-th row, $i = 1, 2, \dots, \alpha$. This Ferrers diagram has then a_j^* dots in the j-th column, $j = 1, 2, \dots, \alpha^*$.

On Fig. 1 we present as an example the Ferrers diagram of the partition $(7, 4, 4, 1)$. It is immediately clear that the partition conjugate to $(7, 4, 4, 1)$ is $(4, 3, 3, 3, 1, 1, 1)$.

Definition. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{P}_N$. If $\sum_{i=1}^r a_i \geq \sum_{i=1}^r b_i$ holds for all values of $r \in \mathbf{N}$, then we write $\boldsymbol{a}\, S\, \boldsymbol{b}$ and say that $\boldsymbol{a}$ is S-greater than $\boldsymbol{b}$.

If neither $\boldsymbol{a}\, S\, \boldsymbol{b}$ nor $\boldsymbol{b}\, S\, \boldsymbol{a}$, then the partitions $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be S-incomparable. S-incomparable partitions exist in $\mathbf{P}_N$, $N \geq 6$.

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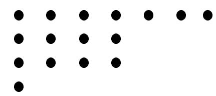


$(7, 4, 4, 1)$

Fig. 1

If $\boldsymbol{a}$ is S-greater than $\boldsymbol{b}$, then the Ferrers diagram of $\boldsymbol{a}$ can be obtained from the Ferrers diagram of $\boldsymbol{b}$ by moving some dots upwards [**4**].

The relation S induces a partaial ordering of the set $\mathbf{P}_N$. Furthermore, $\langle \mathbf{P}_N; S \rangle$ is a lattice. This lattice has been introduced and examined by Snapper [**5**] and somewhat later by Ruch [**3**].

In [**4**] the following result has been proved.

Lemma 1. *If $\boldsymbol{a}$ is a graphic partition and $\boldsymbol{a}\, S\, \boldsymbol{b}$, then $\boldsymbol{b}$ is a graphic partition too.*

A proper consequence of Lemma 1 is that some graphic partitions are maximal with respect to the relation S. Maximal graphic partitions are necessarily mutually S-incomparable. Their structure is determined by the below lemma [**4**].

Let $\boldsymbol{a} = (a_1, a_2, \dots, a_\alpha)$ be a partition of order m, such that $a_1 > a_2 > \cdots > a_\alpha > 0$. Associate to $\boldsymbol{a}$ another partition $\boldsymbol{g} = \boldsymbol{g}[\boldsymbol{a}] = (g_1, g_2, \dots, g_n)$ via $g_j = a_j + j - 1$ and $g_j^* = a_j + 1$, $j = 1, 2, \dots, \alpha$. Note that $n = a_1 + 1$.

Lemma 2. *If $\boldsymbol{a}$ is a partition of the integer m into unequal parts, then $\boldsymbol{g}[\boldsymbol{a}] \in \mathbf{P}_{2m}$ and $\boldsymbol{g}[\boldsymbol{a}]$ is a maximal graphic partition. All maximal graphic partitions in $\mathbf{P}_{2m}$ are of the form $\boldsymbol{g}[\boldsymbol{a}]$.*

According to Lemma 2 the number of maximal graphic partitions of order $2m$ is equal to the number of partitions of m into unequal parts.

In Fig. 2 are presented the Ferrers diagrams of the six only possible maximal graphic partitions of order 16.

In [**4**] it has also been shown that the graph having a vertex degree sequence $\boldsymbol{g}[\boldsymbol{a}]$ is unique. This graph is connected and has cyclomatic number $c = m - a_1 = a_2 + \cdots + a_\alpha$. (Recall that the cyclomatic number of a connected graph with n vertices and m edges is equal to $m - n + 1$.)

Two auxiliary results

Lemma 3. *Let the partition $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ of order $2m$ be graphic. Then there exists a graph G with vertex degree sequence $\boldsymbol{p}$, such that* (a) *G is connected if $m \geq n - 1$,* (b) *G has $n - m$ components if $m \leq n - 1$.*

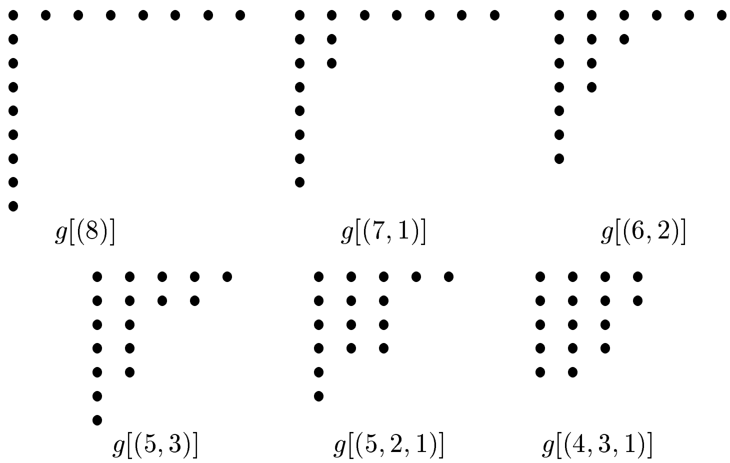
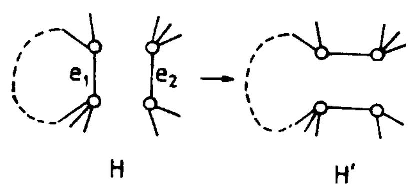


Fig. 2

Proof. Assume first that $m \geq n-1$ and H is a disconnected graph with degree sequence $\boldsymbol{p}$. Let e_1 and e_2 be edges belonging to two different components of H and let e_1 belong to a cycle. Such edges necessarily exist in a disconnected graph with $m \geq n-1$. Then the transformation $H \to H'$ will not change the degree sequence, but will decrease by one the number of components of H.



If H' is disconnected, we can repeat the procedure until a connected graph is obtained.

The proof for the case $m \leq n-1$ is analogous. $\square$

Lemma 4. *A connected graph with cyclomatic number c, $c > 0$, has at least m_c edges where*

(1) $$m_c = c + 1 + \left[\left(1+\sqrt{8c-7}\right)/2\right].$$

Proof is based on the observation that the graph with cyclomatic number c of the form

(2) $$c = x(x-1)/2 + y, \qquad x \in \mathbf{N}, \quad y \in \{1, 2, \dots, x\}$$

and the least number of edges is the graph $G(x,y)$ obtained by joining $y+1$ vertices of K_{x+1} to the (unique) vertex of K_1. Here K_n denotes the complete graph on n

vertices; note that $G(x,x) = K_{x+2}$. The number of edges of $G(x,y)$ is

$$(3) \qquad m_c = x(x+1)/2 + y + 1, \qquad x \in \mathbf{N}, \quad y \in \{1,2,\ldots,x\}.$$

Eq. (1) is obtained from (2) and (3) by simple arithmetic reasoning. □

For the considerations which follow it is purposeful to extend the definition of the quantity m_c, eq. (1), by $m_c = 1$ for $c = 0$.

The main results

In this section we determine the conditions which a partition $\boldsymbol{p}$ of order $2m$ must satisfy in order to correspond to the degree sequence of a connected graph with given cyclomatic number c. Such a graph has $m - c + 1$ vertices and consequently its degree sequence $\boldsymbol{p}$ must have the property $p_1^* = m - c + 1$.

Denote by $\mathbf{P}_{2m}(c)$ the class of all partitions of the number $2m$ into exactly $m - c + 1$ parts. Hence, if $p \in \mathbf{P}_{2m}(c)$, then $p_1^* = m - c + 1$.

It is easy to verify that $\langle \mathbf{P}_{2m}(c); S \rangle$ is a lattice.

Bearing in mind Lemma 1, it is evident that if $\mathbf{P}_{2m}(c)$ contains graphic partitions, then some of them are maximal with respect to the relation $\mathbf{P}_{2m}(c)$. Denote by $\mathbf{P}_{2m}(c; \max)$ the set of maximal graphic partitions in $\mathbf{P}_{2m}(c)$.

From Lemma 2 it follows that the elements of $\mathbf{P}_{2m}(c; \max)$ are of the form $\boldsymbol{g}\big[(m-c, a_2, \ldots, a_\alpha)\big]$, where $(a_2, \ldots, a_\alpha)$ is a partition of the integer c into unequal parts, $c > 0$. (If $c = 0$ then the unique element of $\mathbf{P}_{2m}(c; \max)$ is $\boldsymbol{g}[(m)]$.)

Because of Lemma 3, if $m \geq m_c$ then every graphic partition in $\mathbf{P}_{2m}(c)$ is a vertex degree sequence of a connected graph.

The above observations can be summarized as follows.

LEMMA 5. *A partition $\boldsymbol{p} \in \mathbf{P}_{2m}(c)$ is a vertex degree sequence of a connected graph with cyclomatic number c if and only if $\boldsymbol{q}\ S\ \boldsymbol{p}$ for some $\boldsymbol{q} \in \mathbf{P}_{2m}(c; \max)$. $\mathbf{P}_{2m}(c; \max)$ is non-empty if $m \geq m_c$.*

With these preparations we are able to prove the main results of the present paper, namely Theorems 1–5.

THEOREM 1. *A partition $(p_1, p_2, \ldots, p_n)$ of order $2m$, $m \geq 1$, is the vertex degree sequence of a tree iff $n = m + 1$.*

THEOREM 2. *A partition $(p_1, p_2, \ldots, p_n)$ of order $2m$ is the vertex degree sequence of a connected unicyclic graph iff $m \geq 3$, $n = m$, $p_1 \leq m - 1$ and $p_1 + p_2 \leq m + 1$.*

THEOREM 3. *A partition $(p_1, p_2, \ldots, p_n)$ of order $2m$ is the vertex degree sequence of a connected bicyclic graph iff $m \geq 5$, $n = m - 1$, $p_1 \leq m - 2$, $p_1 + p_2 \leq m + 1$ and $p_1 + p_2 + p_3 \leq m + 3$.*

THEOREM 4. *A partition* $(p_1, p_2, \dots, p_n)$ *of order* $2m$ *is the vertex degree sequence of a connected tricyclic graph iff* $m \geq 6$, $n = m - 2$, $p_1 \leq m - 3$ *and either* $(p_1 + p_2 \leq m + 1$ *and* $p_1 + p_2 + p_3 \leq m + 3)$ *or* $p_1 + p_2 \leq m$.

THEOREM 5. *A partition* $(p_1, p_2, \dots, p_n)$ *of order* $2m$ *is the vertex degree sequence of a connected tetracyclic graph iff* $m \geq 8$, $n = m - 3$, $p_1 \leq m - 4$ *and either* $(p_1 + p_2 \leq m$ *and* $p_1 + p_2 + p_3 \leq m + 3)$ *or* $p_1 + p_2 \leq m + 1$.

Proofs

The general strategy in proving Theorems 1–5 is the following. From Lemma 2 we know the number and the structure of the elements of $\mathbf{P}_{2m}(c; \max)$. Bearing in mind Lemma 5 we have just to find the condition(s) needed that an element of $\mathbf{P}_{2m}(c)$ is not S-greater than any element of $\mathbf{P}_{2m}(c; \max)$. In other words we have to avoid the partitions whose Ferrers diagrams are obtained by moving upwards a dot of the Ferrers diagram of $\boldsymbol{p} \in \mathbf{P}_{2m}(c; \max)$.

Denote by $[i, j]$ the dot in a Ferrers diagram lying in the i-th row and in the j-th column.

Proof of Theorem 1. Theorem 1 is a well known result [**2**]. We present its proof for reasons of completeness.

The unique element of $\mathbf{P}_{2m}(0; \max)$ is $\boldsymbol{g}[(m)] = (m, 1, 1, \dots, 1)$. In order to construct a partition which is S-greater than $(m, 1, 1, \dots, 1)$ we would have to move the dot $[m + 1, 1]$ of the respective Ferrers diagram into position $[1, m + 1]$. Such a transformation would, however, violate the condition $n = m + 1$.

Hence $(m, 1, 1, \dots, 1)$ is S-greater than any element of $\mathbf{P}_{2m}(0)$ i.e. every element of $\mathbf{P}_{2m}(0)$ is a vertex degree sequence of a tree. □

Proof of Theorem 2. The unique element of $\mathbf{P}_{2m}(1; \max)$ is $\boldsymbol{g}[(m - 1, 1)] = (m - 1, 2, 2, 1, 1, \dots, 1)$. A partition $\boldsymbol{p} \in \mathbf{P}_{2m}(1)$ will become S-greater than $\boldsymbol{g}[(m - 1, 1)]$ if the dot $[3, 2]$ in the respective Ferrers diagram is moved either into the position $[1, m]$ or into the position $[2, 3]$ (c.f. Fig. 2). The transformation $[3, 2] \to [1, m]$ would increase p_1 by one. In order to avoid this we have to require $p_1 \leq m - 1$. The transformation $[3, 2] \to [2, 3]$ would increase p_2 by one, leaving p_1 unchanged. Bearing in mind the definition of the relation S we have to require $p_1 + p_2 \leq (m - 1) + (2)$. This immediately yields Theorem 2. □

Proof of Theorem 3 is analogous thanks to the fact that $\mathbf{P}_{2m}(2; \max)$ also has a unique element $\boldsymbol{g}[(m - 2, 2)] = (m - 2, 3, 2, 2, 1, 1, \dots, 1)$.

Proof of Theorem 4. $\mathbf{P}_{2m}(3; \max)$ has two elements: $\boldsymbol{g}[(m - 3, 3)]$ and $\boldsymbol{g}[(m - 3, 2, 1)]$ (c.f. Fig. 2). In order to obtain a partition $\boldsymbol{p} \in \mathbf{P}_{2m}(3)$ which is S-greater than $\boldsymbol{g}[(m - 3, 3)]$ we have to make one of the following three transformations:

(a) $[5, 1] \to [1, m - 2]$ or $[2, 4] \to [1, m - 2]$; (b) $[5, 1] \to [2, 5]$; (c) $[5, 1] \to [3, 3]$.

In order to avoid (a) we have to require $p_1 \leq m-3$. In order to avoid (b) we have to require $p_1+p_2 \leq (m-3)+(4)$. In order to avoid (c) we have to require $p_1+p_2+p_3 \leq (m-3)+(4)+(2)$.

In an analogous manner the conditions that $\boldsymbol{p} \in \mathbf{P}_{2m}(3)$ is not S-greater than $\boldsymbol{g}[(m-3,2,1)] = (m-3,3,3,3,1,1,\dots,1)$ are $p_1 \leq m-3$ and $p_1+p_2 \leq (m-3)+(3)$.

All these conditions together result in Theorem 4. $\square$

Proof of Theorem 5 is analogous to the proof of Theorem 4 since also $\mathbf{P}_{2m}(4;\max)$ has two elements.

From the above proofs it is evident that by continuing a similar way of reasoning and applying Lemmas 2 and 5 one can characterize the vertex degree sequences of connected graphs with cyclomatic numbers $c \geq 5$. These characterizations are somewhat more complicated because for $c \geq 5$, $|\mathbf{P}_{2m}(c;\max)| \geq 3$. A typical result of this kind is Theorem 6 which we state without proof.

THEOREM 6. *A partition* $(p_1,p_2,\dots,p_n)$ *of order* $2m$ *is the vertex degree sequence of a connected pentacyclic graph iff* $m \geq 9$, $n = m-4$ *and either* (a) *or* (b) *or* (c) *holds*:

(a) $p_1+p_2 \leq m+1$.

(b) $p_1+p_2 \leq m$; $\quad p_1+p_2+p_3 \leq m+3$; $\quad p_1+p_2+p_3+p_4 \leq m+6$; $p_1+p_2+p_3+p_4+p_5 \leq m+8$.

(c) $p_1+p_2 \leq m-1$; $\quad p_1+p_2+p_3 \leq m+3$; $\quad p_1+p_2+p_3+p_4 \leq m+6$.

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