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ALL GENERAL REPRODUCTIVE SOLUTIONS OF BOOLEAN EQUATIONS

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Abstract. An equation is called a finite equation if its solution set is a subset of a given finite set. Applying some kind of algebraic structure S. Prešić determined in [**7**] the formulas of all general reproductive solutions of such equations. Using Prešić's result we determine in this paper the formulas of all general reproductive solutions of arbitrary Boolean equations.

Definition 1. Let E be a given non-empty set and eq be a given unary relation of E. A formula $x = \varphi(p)$, where $\varphi: E \to E$ is a given function, represents a general reproductive solution of the x-equation $\mathrm{eq}(x)$ if and only if

$$(\forall t)\,\mathrm{eq}\big(\varphi(t)\big) \wedge (\forall t)\big(\mathrm{eq}(t) \Rightarrow t = \varphi(t)\big).$$

Definition 2. Let $Q = (q_0, q_1, \dots, q_m)$ be a given set of $m+1$ elements and $S = \{0, 1\}$. Define the operation x^y in the following way:

$$x^y = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases} \qquad (x, y \in Q \cup S).$$

The operations $+$ and $\circ$ are described by the following tables

$+$	0	1
0	0	1
1	1	1

$\circ$	0	1
0	0	0
1	0	1

Extending these operations to the partial operations of the set $Q \cup S$ by putting

$$x + 0 = x,\ 0 + x = x,\ x \circ 0 = 0,\ 0 \circ x = 0, x \circ 1 = x,\ 1 \circ x = x,$$

S. Prešić considered the following x-equation

$$s_0 \circ x^{q_0} + s_1 \circ x^{q_1} + \cdots + s_m \circ x^{q_m} = 0, \tag{1}$$

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where $s_i \in (0,1)$ are given elements and $x \in Q$ is unknown. It is obvious that the equation (1) is consistent if and only if $s_0 \circ s_1 \circ \cdots \circ s_m = 0$. The set $Z(a_0, \dots, a_m)$, where $(a_0, \dots, a_m) \in S^{m+1}$, is defined as follows:

$$q_i \in Z(a_0, \dots, a_m) \Leftrightarrow a_i = 0 \quad (i = 0, 1, \dots, m).$$

THEOREM 1. (Prešić). *If* (1) *is a consistent equation then the formula* $x = A(t)$, (t *is any element of* Q) *represents a general reproductive solution of the equation* (1) *if and only if*

$$A(x) = C_0(s_0, \dots, s_m)x^{q_0} + \cdots + C_m(s_0, \dots, s_m)x^{q_m}$$

where each coefficient C_k *is determined by some equality of the form*

$$C_k(s_0, \dots, s_m) = b_k s_k^0 + \sum_{a_k=1,\, a_0 \dots a_m = 0} F_k(a_0, \dots, a_m) s^{a_0} \dots s^{a_m}$$

assuming that the coefficients $F_k(a_0, \dots, a_m) \in Q$ *satisfy the following condition*

$$F_k(a_0, \dots, a_m) \in Z(a_0, \dots, a_m) \quad (\textit{we omit } \circ).$$

Let $X = (x_1, \dots, x_n) \in B^n$ and $T = (t_1, \dots, t_n) \in B^n$, where $(B, \cup, \cdot, ', 0, 1)$ is a Boolean algebra. Let $p = 2^n - 1$, $\{0,1\}^n = (A_0, \dots, A_m)$ and $A_j = \big((A_j)_1, \dots, (A_j)_n\big)$ $(j = 0, 1, \dots, p)$.

THEOREM 2. (c.f. [**8**]) *Let* $f\colon B^n \to B$ *be a Boolean function. The equation* $f(X) = 0$ *is consistent if and only if* $\prod_{i=0}^{p} f(A_i) = 0$.

THEOREM 3. (Deschamps). *Let* $f, g_1, \dots, g_n\colon B^n \to B$ *be Boolean functions and* $G = (g_1, \dots, g_n)$. *The formulas*

$$x_j = g_j(t_1, \dots, t_n) \quad (j = 1, \dots, n) \tag{2}$$

i.e. $X = G(T)$, *define a general reproductive solution of the consistent Boolean equation* $f(X) = 0$ *if and only if*

$$(\forall T)\left(f\big(G(T)\big) = 0 \wedge f(T) = \bigcup_{j=1}^{n} \big(g_j(T) \oplus t_j\big)\right)$$

where $a \oplus b = a'b \cup ab'$.

COROLLARY. *The formulas* (2) *define a general reproductive solution of* $f(X) = 0$ *if and only if*

$$\bigcup_{i=0}^{p}\left(f\big(G(A_i)\big) \cup f(A_i) \oplus \bigcup_{j=1}^{n}\big(g_j(A_j) \oplus (A_i)_j\big)\right) = 0.$$

The proof of Corollary follows from:

$$(\forall a, b \in B)(a = 0 \wedge b = 0 \Leftrightarrow a \cup b = 0) \tag{3}$$

$$(\forall a, b \in B)(a = b \Leftrightarrow a \oplus b = 0) \tag{4}$$

$$(\forall T) h(T) = 0 \Leftrightarrow \bigcup_{i=0}^{p} h(A_i) = 0, \tag{5}$$

where $h: B^n \to B$ is an arbitrary Boolean function.

Definition 3. The *Horn formulas over language* L are defined as follows:

— the *elementary Horn formulas* are the atomic formulas of L and the formulas of the form $D_1 \wedge \ldots \wedge D_s \Rightarrow D$, where $D_1, \ldots, D_s, D$ are atomic;

— every Horn formula is built from elementry Horn formulas using $\wedge$, $\forall$, $\exists$.

THEOREM 4 (Vaught). *Let H be a Horn sentence (formula without free variables) in the language L_B of Boolean algebras. If $B_2 \vdash H$ then $B \vdash H$.*

See, for instance, [**4**].

Let $A_i \in W(a_0, \ldots, a_p) \Leftrightarrow a_i = 0 \quad (i = 0, \ldots, p)$.

THEOREM 5. *Let $f, g_1, \ldots, g_n: B^n \to B$, be Boolean functions and $G = (g_1, \ldots, g_n)$. If the equation $f(X) = 0$ is consistent then the formula $X = G(T)$ represents a general reproductive solution of $f(X) = 0$ if and only if $G(T)$ can be writen in the form*

(6)

$$G(T) = \bigcup_{k=0}^{p} \Big[A_k f'(A_k) \cup \bigcup_{a_k=1, a_0 \ldots a_p=0} F_k((a_0, \ldots, a_p)\big(f(A_0)^{a_0} \ldots f(A_p)\big)^{a_p} \Big] T^{A_i}$$

assuming that

$$\big(\forall k \in (0, \ldots, p)\big)\big(\forall a_0, \ldots, a_p \in \{0,1\}\big)\big(a_k = 1 \wedge a_0 \ldots a_p = 0 \\ \Longrightarrow F_k(a_0, \ldots, a_p) \in W(a_0, \ldots, a_p)\big).$$

Proof. Let F be a $(p+1)(2^p - 1)$-tuple of all elements $F_k(a_0, \ldots, a_p)$ occuring in (6). Bearing in mind Theorem 2 and Corollary we can write Theorem 5 as follows

$$(\forall f)(\forall G)\Big[\prod_{i=0}^{p} f(A_i) = 0 \\ \Rightarrow \Big[\bigcup_{i=0}^{p}\Big(f(G(A_i)) \cup \Big(f(A_i) \oplus \bigcup_{j=1}^{n}(g_j(A_i) \oplus (A_i)_j)\Big)\Big) = 0\Big] \\ \Longleftrightarrow (\exists F)\Big[(\forall T) G(T) = \bigcup_{k=0}^{p}\Big(A_k f'(A_k)$$

$$\cup \bigcup_{a_k=1,\, a_0\dots a_p=0} F_k(a_0,\dots,a_p)(f(A_0))^{a_0}\dots(f(A_p))^{a_p}\Big)T^{A_k}$$

$$\wedge\,(\forall k\in\{0,\dots,1\})(\forall a_0,\dots,a_p\in\{0,1\})\Big(a_k=1\wedge a_0\dots a_p=0$$

$$\Rightarrow F_k(a_0,\dots,a_p)\in W(a_0,\dots,a_p)\Big)\Big]\Big].$$

The formula

$$(\forall T)G(T)=\bigcup_{k=0}^{p}\Big(A_k f'(A_k)$$

$$\cup \bigcup_{a_k=1,\, a_0\dots a_p=0} F_k(a_0,\dots,a_p)\big(f(A_0)\big)^{a_0}\dots\big(f(A_p)\big)^{a_p}\Big)T^{A_k}$$

can be written as

$$\bigcup_{i=0}^{p}\bigcup_{j=1}^{n}\Big(g_j(A_i)\oplus\bigcup_{k=0}^{p}\Big((A_k)_j f'(A_k)$$

$$\cup \bigcup_{a_k=1,\, a_0\dots a_p=0}\big(F_k(a_0,\dots,a_p)\big)_j\big(f(A_0)\big)^{a_0}\dots\big(f(A_p)\big)^{a_p}A_i^{A_k}\Big)\Big)=0$$

because of (3) and (4). The formula $F_k(a_0,\dots,a_p)\in W(a_0,\dots,a_p)$ can be written as

$$(\exists T)F_k(a_0,\dots,a_p)=A_{j_0(a_0,\dots,a_p)}T^{A_0}\cup\dots\cup A_{j_p(a_0,\dots,a_p)}T^{A_p}$$

i.e.

$$\prod_{i=0}^{p}\Big(F_k(a_0,\dots,a_p)\oplus(A_{j_0(a_0,\dots,a_p)}A_i^{A_0}\cup\dots\cup A_{j_p(a_0,\dots,a_p)}A_i^{A_p})\Big)=0$$

where $(A_{j_0(a_0,\dots,a_p)},\dots,A_{j_p(a_0,\dots,a_p)})=W(a_0,\dots,a_p)$. Therefore we can write the formula

$$\big(\forall k\in\{0,\dots,p\}\big)\big(\forall a_0,\dots,a_p\in\{0.1\}\big)\,(a_k=1\wedge a_0\dots a_p=0)$$
$$\Longrightarrow\ (F_k(a_0,\dots,a_p)\in W(a_0,\dots,a_p))$$

as follows

$$\bigcup_{k=0}^{p}\ \bigcup_{a_k=1,a_0\dots a_p=0}\ \prod_{i=0}^{p}\big(F_k(a_0,\dots,a_p)\oplus(A_{j_0}(a_0,\dots,a_p)A_i^{A_0}\cup\dots$$

$$\cup A_{j_p}(a_0,\dots,a_p)A_i^{A_p})\big)=0.$$

Let

$$\big(\forall i \in \{0,\ldots,p\}\big) y_i = f(A_i)$$
$$\big(\forall j \in \{1,\ldots,n\}\big)\big(\forall i \in \{0,\ldots,p\}\big) g_{j,i} = g_j(A_i)$$
$$f(X) = \bigcup_{i=0}^{p} y_i X^{A_i} \quad \text{and} \quad g_j(X) = \bigcup_{i=0}^{p} g_{j,i} X^{A_i} \quad (j = 1,\ldots,n).$$

We also introduce the following notation: $Y = (y_0,\ldots,y_p)$ and

$$G_{n\times p} = \begin{bmatrix} g_{1,0} & \cdots & g_{1,p} \\ \vdots & & \vdots \\ g_{n,0} & \cdots & g_{n,p} \end{bmatrix}$$

Then Theorem 5 can be written as

$$\begin{aligned}
&(\forall Y)(\forall G_{n\times p})\Bigg(\prod_{i=0}^{p} y_i = 0 \\
&\quad \Longrightarrow \Bigg(\Bigg(\bigcup_{i=0}^{p}\Bigg(\bigcup_{k=0}^{p} y_k (g_{1,i},\ldots,g_{n,i})^{A_k} \cup \Bigg(y_i \oplus \bigcup_{j=1}^{n} (g_{j,i} \oplus (A_i)_j)\Bigg)\Bigg) = 0\Bigg) \\
&\qquad \Longleftrightarrow (\exists F)\Bigg(\bigcup_{i=0}^{p}\bigcup_{j=1}^{p}\Bigg(g_{j,i} \oplus \bigcup_{k=0}^{p}\Bigg((A_k)_j y_k' \\
&\qquad \oplus \bigcup_{a_k=1,\, a_0\ldots a_p=0} (F_k(a_0,\ldots,a_p))_j y_0^{a_0}\ldots y_p^{a_p}\Bigg) A_i^{A_k} \\
&\qquad \cup \bigcup_{k=0}^{p} \bigcup_{a_k=1,\, a_0\ldots a_p=0} \prod_{i=0}^{p}\Big(F_k(a_0,\ldots,a_p) \\
&\qquad \oplus \Big(A_{j_0(a_0,\ldots,a_p)} A_i^{A_0} \cup \ldots \cup A_{j_p(a_0,\ldots,a_p)} A_i^{A_p}\Big)\Big) = 0\Bigg)\Bigg)\Bigg)
\end{aligned}$$

i.e.

$$\begin{aligned}
&(\forall Y)(\forall G_{n\times p})\Bigg(\prod_{i=0}^{p} y_i = 0 \\
&\quad \Longrightarrow \Bigg(\Bigg(\bigcup_{i=0}^{p}\Bigg(\bigcup_{k=0}^{p} y_k (g_{1,i},\ldots,g_{n,i})^{A_k} \cup \Bigg(y_i \oplus \bigcup_{j=1}^{n} (g_{j,i} \oplus (A_i)_j)\Bigg)\Bigg) = 0\Bigg) \\
&\qquad \Longleftrightarrow \prod_{F}\Bigg(\bigcup_{i=0}^{p}\bigcup_{j=1}^{p}\Bigg(g_{j,i} \oplus \bigcup_{k=0}^{p}\Bigg((A_k)_j y_k' \\
&\qquad \oplus \bigcup_{a_k=1,\, a_0\ldots a_p=0} (F_k(a_0,\ldots,a_p))_j y_0^{a_0}\ldots y_p^{a_p}\Bigg) A_i^{A_k} = 0
\end{aligned}$$

$$\wedge \bigcup_{k=0}^{p} \bigcup_{a_k=1,\, a_0\dots a_p=0} \prod_{i=0}^{p} \Big(F_k(a_0,\dots,a_p) \oplus \Big(A_{j_0(a_0,\dots,a_p)} A_i^{A_0} \cup \dots \cup A_{j_p(a_0,\dots,a_p)} A_i^{A_p} \Big) \Big) = 0 \Big) \Big) \Big)$$

where $\prod_F$ denotes the product over all $F \in \big(\{0,1\}^n\big)^{(p+1)(2^p-1)}$. The last formula is a Horn sentence because

$$r_1 \Rightarrow (r_2 \Leftrightarrow r_3) \Leftrightarrow (r_1 \wedge r_2 \Rightarrow r_3) \wedge (r_1 \wedge r_3 \Rightarrow r_2)$$

is a tautology. Therefore it is sufficient to prove Theorem 5 in B_2 bearing in mind Theorem 4. The proof of Theorem 5 in B_2 immediately follows from Prešić's Theorem 1 assuming that $Q = \{0,1\}^n$, $x = X$, $p = T$, $(s_0,\dots,s_p) = \big(f(A_0),\dots,f(A_p)\big)$, $Z = W$, $+$ and $\circ$ are $\cup$ and $\cap$, respectively.

Example. Let us determine the formulas of all general reproductive solutions of the general Boolean equation in two unknowns

$$ax'y' \cup bx'y \cup cxy' \cup dxy = 0. \tag{6}$$

Let $A_0 = (0,0)$, $A_1 = (0,1)$, $A_2 = (1,0)$ and $A_3 = (1,1)$. In accordance with Theorem 5 all reproductive general solutions of (6) can be represented by

$$\begin{aligned}(x,y) = &\Big[(0,0)a' \cup \bigcup_{a_0=1,\, a_0a_1a_2a_3=0} F_0(a_0,a_1,a_2,a_3) a^{a_0} b^{a_1} c^{a_2} d^{a_3} \Big] t_1' t_2',\\ &\cup \Big[(0,1)b' \cup \bigcup_{a_1=1,\, a_0a_1a_2a_3=0} F_1(a_0,a_1,a_2,a_3) a^{a_0} b^{a_1} c^{a_2} d^{a_3} \Big] t_1' t_2\\ &\cup \Big[(1,0)c' \cup \bigcup_{a_2=1,\, a_0a_1a_2a_3=0} F_2(a_0,a_1,a_2,a_3) a^{a_0} b^{a_1} c^{a_2} d^{a_3} \Big] t_1 t_2'\\ &\cup \Big[(1,1)d' \cup \bigcup_{a_3=1,\, a_0a_1a_2a_3=0} F_3(a_0,a_1,a_2,a_3) a^{a_0} b^{a_1} c^{a_2} d^{a_3} \Big] t_1 t_2,\end{aligned}$$

where $F_i(a_0,a_1,a_2,a_3) \in W(a_0,a_1,a_2,a_3) \quad (i = 0,1,2,3)$ and

$$\begin{aligned}
&W(1,0,0,0) = \{(0,1),(1,0),(1,1)\}, && W(1,0,1,1) = \{(0,1),(1,0)\},\\
&W(1,0,1,0) = \{(0,1),(1,1)\}, && W(1,0,1,1) = \{(0,1)\},\\
&W(1,1,0,0) = \{(1,0),(1,1)\}, && W(1,1,0,1) = \{(1,0)\},\\
&W(1,1,1,0) = \{(1,1)\}, && W(0,0,0,1) = \{(0,0),(0,1),(1,0)\}\\
&W(0,0,1,0) = \{(0,0),(0,1),(1,1)\}, && W(0,0,1,1) = \{(0,0),(0,1)\},\\
&W(0,1,0,0) = \{(0,0),(1,0),(1,1)\}, && W(0,1,0,1) = \{(0,0),(1,0)\}\\
&W(0,1,1,0) = \{(0,0),(1,1)\}, && W(0,1,1,1) = \{(0,0)\}.
\end{aligned}$$

If $F_i(a_0, a_1, a_2, a_3)$ is, for instance, the element written first in the set $W(a_0, a_1, a_2, a_3)$, we get

$$\begin{aligned}(x,y) = [&(1,1)a' \cup (0,0)ab'c'd' \cup (0,1)ab'c'd \cup (0,1)ab'cd' \\ &\cup (0,1)ab'cd \cup (1,0)abc'd' \cup (1,0)abc'd \cup (1,1)abcd']\, t_1t_2 \\ &[(0,1)b' \cup (0,0)a'bc'd' \cup (0,0)a'bc'd \cup (0,0)a'bcd' \\ &\cup (0,0)a'bcd \cup (1,0)abc'd' \cup (1,0)abc'd \cup (1,1)abcd']\, t_1't_2 \\ &[(1,0)c' \cup (0,0)a'b'cd' \cup (0,0)a'b'cd \cup (0,0)a'bcd' \\ &\cup (0,0)abcd \cup (0,1)ab'cd' \cup (0,1)ab'cd \cup (1,1)abcd']\, t_1t_2' \\ &[(1,1)d' \cup (0,0)a'b'c'd \cup (0,0)a'b'cd \cup (0,0)a'bc'd \\ &\cup (0,0)a'bcd \cup (0,1)ab'c'd \cup (0,1)ab'cd \cup (1,0)abc'd]\, t_1t_2\end{aligned}$$

i.e.

$$\begin{aligned}x &= (abc'd' \cup abc'd \cup abcd')t'_1t_2 \cup (abc'd' \cup abc'd \cup abcd')t'_1t_2 \\ &\quad \cup (c' \cup abcd')t_1t'_2 \cup (d' \cup abc'd)t_1t_2 \\ y &= (ab'c'd' \cup ab'c'd \cup ab'cd' \cup ab'cd \cup abcd')t'_1t'_2 \cup (b' \cup abcd')t'_1t_2 \\ &\quad \cup (ab'cd' \cup ab'cd \cup abcd')t_1t'_2 \cup (d' \cup ab'c'd \cup ab'cd)t_1t_2\end{aligned}$$

i.e.

$$\begin{aligned}x &= ab(c' \cup d')t'_1t'_2 \cup ab(c' \cup d)t'_1t_2 \cup (c' \cup abd')t_1t'_2 \cup (d' \cup abc')t_1t_2 \\ y &= a(b' \cup cd')t'_1t'_2 \cup (b' \cup acd')t'_1t_2 \cup ac(b' \cup d')t_1t'_2 \cup (d' \cup ab')t_1t_2.\end{aligned}$$

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