PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 46 (60), 1989, 20–24

ON THE QUOTIENT (Ω, m)-RINGOIDS

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Abstract. Universal algebras considered in this paper are generalizations of rings, distributive lattices, semirings and composite rings. We consider the quotient ring construction of (Ω, m)-ringoid and extend a result of Crombez and Timm about (n, m)-rings.

1. Introduction. An algebra A of a variety V is said to be *simple* if its congruence lattice $\mathrm{Con}(A)$ is isomorphic to the two-element chain. A variety V of algebras is said to have the *simple extension property* if any algebra A in V can be embedded into a simple algebra B in V. For each integer $m \geq 2$, we denote by $\mathcal{G}(m)$ the variety of all m-groupoids, i.e. algebras with m-ary operation. It was shown in [**5**] that $\mathcal{G}(m)$ has the simple extension property.

However, there exist many varieties of algebras which do not have the simple extension property for example, the variety of semilattices and the variety of associative p^k-rings. If V is such a variety then it is natural to investigate what kind of algebras in V can be embedded in simple algebras. For the variety of commutative rings, Grell [**4**] introduced the construction of quotient rings of commutative rings in 1927. The classical result in ring theory states that the quotient ring of an integral domain is a simple ring.

In [**2**], Crombez introduced a variety of algebras of type $\langle n, m\rangle$ which he called the (n, m)-rings. If $n = m = 2$, the $(2, 2)$-rings are the ordinary associative rings. In [**3**], Crombez and Timm generalized the usual concept of integral domain and field in ring theory to (n, m)-rings and they showed that any commutative (n, m)-integral domain can be embedded into an (n, m)-field.

The purpose of this paper is to introduce a variety of algebras which we call (Ω, m)-*ringoids*. The (Ω, m)-ringoids are a generalization of (n, m)-rings, distributive lattices, semirings and commutative composite rings of Irving Adler [**1**], etc. We show that the result of Crombez and Timm can be generalized to (Ω, m)-ringoids. However, an example is shown that the (Ω, m)-field need not be simple.

AMS Subject Classification (1980): Primary 08 A 05

Key-words and phrases: ringoid, m-semigroups, congruence, integral domain, simple, lattices and composite ring.

2. Definitions and Examples. Let $m > 1$ be an integer. We recall the following concept (see [**6**]).

Definition 2.1. An m-groupoid $A = \langle A, [\ ]\rangle$ is called an *m-semigroup* if it satisfies the following generalized associative law:

$$\begin{aligned}[][x_1x_2\dots x_n]x_{n+1}\dots x_{2n-1}] &= [x_1[x_2x_3\dots x_{n+1}]x_{n+2}\dots x_{2n-1}]\\ &= \dots\\ &= [x_1x_2\dots x_{n-1}[x_nx_{n+1}\dots x_{2n-1}]].\end{aligned}$$

For simplicity, we shall denote the element $[x_1\dots x_m]$ simply by $x_1\dots x_n$.

Definition 2.2. A universal aglebra $\langle A, \Omega, [\ ]\rangle$ is called an *(Ω, m)-ringoid* if

(1) $\langle A, [\ ]\rangle$ is an m-semigroup,

(2) Ω is a set of operations such that each of them has arity greater than one,

(3) $[\ ]$ is distributive with resect to every w in Ω i.e. if w is an n-ary operation then for any $a_2, a_3, \dots, a_m, b_1, b_2, \dots, b_n$ of elements of A, we have

$$\begin{gathered}a_2a_3\dots a_i\, w\,(b_1,\dots,b_n)a_{i+1}\dots a_m\\ = w(a_2\dots a_ib_1a_{i+1}\dots a_m, a_2\dots a_ib_2a_{i+1}\dots a_m,\dots, a_2\dots a_ib_na_{i+1}\dots a_m)\end{gathered}$$

for each $i = 1, 2, \dots, m$.

Definition 2.3. An (Ω, m)-ringoid is said to be *commutative* if for each $x_1,\dots,x_m \in A$ and any permutation $\sigma \in S(m)$ of $\{1,\dots,m\}$ we have $x_1\dots x_m = x_{\sigma(1)}\dots x_{\sigma(m)}$, i.e. $\langle A; [\ ]\rangle$ is a commutative m-semigroup.

Definition 2.4. Adler [**1**] called the following algebra $\langle R, +, \circ\rangle$ a *composition ring*: (1) $\langle R, +, \circ\rangle$ is a commutative ring; (2) $\langle R, \circ\rangle$ is a semigroup such that $(f+g)\circ h = f\circ h + g\circ h$, $(f\times g)\circ h = f\circ h\times g\circ h$, for all f, g, h in R.

If $\langle R, \circ\rangle$ is commutative then we have a commutative $(\{+, \times\}, 2)$-ringoid.

Example 2.5. The (n, m)-ring $\langle A; w, [\ ]\rangle$ of Crombez [**2**] is an (Ω, m)-ringoid where $\Omega = \{w\}$ and

(1) $\langle A; w\rangle$ is commutative n-semigroup;

(2) for each $g_1,\dots,g_{i-1}, g_{i+1},\dots,g_n, h \in A$, a unique solution in the indeterminate x_i exists for the equation $w(g_1,\dots,g_{i-1}, x_i, g_{i+1},\dots,g_n) = h$, for each $i = 1,\dots,n$.

For a more general theory of (n, m)-rings see [**7**] and [**8**].

Example 2.6. Any semiring $\langle A, +, \circ\rangle$ is an $(\Omega, 2)$-ringoid where $\Omega = \{+\}$. In particular, the distributive lattice $\langle L; \vee, \wedge\rangle$ is a $(\{\vee\}, 2)$-ringoid.

Example 2.7. Let $\langle A; +\rangle$ be a group. Define a ternary operation w on A as follows: $w(x, y, z) = x - y + z$; then $\langle A; w, +\rangle$ is an $(\Omega, 2)$-ringoid where $\Omega = \{w\}$. For we have

$$w(x+a, y+a, z+a) = (x+a) - (y+a) + (z+a) = x - y + z + a = w(x, y, z) + a$$

and also $w(a+x, a+y, a+z) = a + w(x, y, z)$ for any $x, y, z, a \in A$.

Definition 2.8. A zero of (Ω, m)-ringoid is an element z such that $zx_1 \dots x_{m-1} = x_1 z \dots x_{m-1} = \dots = x_1 x_2 \dots x_{m-1} z = z$ for all x_i in A.

If a zero exists, it is unique and denoted by 0. Put $R^* = R \setminus \{0\}$ if 0 exists and $R^* = R$ otherwise.

Example 2.9. If $\langle R, +, \circ \rangle$ is a $(2,2)$-ring we define $w(x, y, z) = x + y + z$; then $\langle R; w, +, \circ \rangle$ is an $(\Omega, 2)$-ring with a zero.

Example 2.10. Let R be a set of all odd integers. We define a ternary operation $w(x, y, z) = x + y + z$ and a binary operation $x \circ y = x \times y$; then $\langle R; w, \circ \rangle$ is a $(\{w\}, 2)$-ringoid without a zero.

Example 2.11. An m-semigroup $\langle A; [\,] \rangle$ is called *right* (resp. *left*) *cancellative with respect to* $M \subseteq A$, if for $b_j \in M$ and a_i, c_i in A, we have

$$a_1 b_2 \dots b_m = c_1 b_2 \dots b_m \quad \text{implies} \quad a_1 = c_1, \tag{1}$$

resp.

$$b_1 \dots b_{m-1} a_m = b_1 \dots b_{m-1} c_m \quad \text{implies} \quad a_m = c_m. \tag{2}$$

Crombez and Timm [**3**] showed that a right and left cancellative semigroup (with respect to M) is cancellative (with respect to M) i.e. for $i = 1, \dots, m$, we have

$$b_1 \dots b_{i-1} a_i b_{i+1} \dots b_m = b_1 \dots b_{i-1} c_i b_{i+1} \dots b_m \quad \text{implies} \quad a_i = c_i.$$

3. A construction of quotient (Ω, m)-ringoids. Let $\langle A, \Omega, [\,] \rangle$ be a commutative (Ω, m)-ringoid such that $\langle A; [\,] \rangle$ is cancellative with respect to a (non-empty) m-subsemigroup S. We define a relation $\equiv$ on $A \times S^{m-1}$ as follows:

$$(a, s_2, \dots, s_m) \equiv (b, s'_2, \dots, s'_m) \quad \text{if} \quad a s'_2 \dots s'_m = b s_2 \dots s_m.$$

LEMMA 3.12. *The relation $\equiv$ is an equivalence relation.*

Proof. Clearly $\equiv$ is reflexive and symmetric. To see that $\equiv$ is transitive, we let $(a, s_2, \dots, s_m) \equiv (b, s'_2, \dots, s'_m)$ and $(b, s'_2, \dots, s'_m) \equiv (c, s''_2, \dots, s''_m)$; then we have $a s'_2 \dots s'_m = b s_2 \dots s_m$ and $b s''_2 \dots s''_m = c s'_2 \dots s'_m$.

Hence $a s''_2 \dots s''_m s'_2 \dots s'_m = a s'_2 \dots s'_m s''_2 \dots s''_m = b s_2 \dots s_m s''_2 \dots s''_m$.

Therefore we have $a s''_2 \dots s''_m = c s_2 \dots s_m$, which shows that $(a, s_2, \dots, s_m) = (c, s''_2, \dots, s''_m)$. ∎

We shall denote the equivalence class of $A_S = A \times S^{m-1} / \equiv$ which contains $(a, s_2, \dots, s_m)$ by $\dfrac{a}{s_2 \dots s_m}$.

For each n-ary $w \in \Omega$ we define

$$w\left(\frac{a_1}{s_{12}\dots s_{1m}}, \dots, \frac{a_n}{s_{n2}\dots s_{nm}}\right) = \frac{w(a_1 s_{22}\dots s_{nm}, \dots, a_n s_{12}\dots s_{1m}\dots s_{(n-1)2}\dots s_{(n-1)m})}{s_{12}\dots s_{1m}\dots s_{n2}\dots s_{nm}},$$

and

$$\left[\frac{a_1}{s_{12}\dots s_{1m}}, \dots, \frac{a_m}{s_{m2}\dots s_{mm}}\right] = \frac{a_1\dots a_m}{s_{12}\dots s_{1m}\dots s_{m2}\dots s_{mm}}.$$

We have

Theorem 3.13. *The algebra $\langle A_S; \Omega, [\,]\rangle$ is an (Ω, m)-ringoid. If A has a zero then A_S has a zero and the map $\mu : A \mapsto A_S$ defined by $\mu(a) = \dfrac{as_2\dots s_m}{s_2\dots s_m}$ is an injective homomorphism.*

We shall call $\langle A_S; \Omega, [\,]\rangle$ the *quotient* (Ω, m)*-ringoid* of A with respect to S.

A commutative (Ω, m)-ringoid is called a *generalized integral domain* if it is cancellative. If for each $w \in \Omega$ the n-groupoid $\langle A; w\rangle$ is a commutative n-group and the (Ω, m)-ringoid is a generalized integral domain then it is called an (Ω, m)*-integral domain.*

Theorem 3.15. *Let $\langle A; \Omega, [\,]\rangle$ be a commutative (Ω, m)-ringoid such that $\langle A; [\,]\rangle$ is cancellative with respect to an m-subsemigroup S. Then $\langle A_S; [\,]\rangle$ is cancellative with respect to its m-subgroup $S_S = \left\{\dfrac{s_1}{s_2\dots s_m} : s_i \in S\right\}$.*

Theorem 3.16. *Let $\langle A; \Omega, [\,]\rangle$ be an (Ω, m)-ringoid with zero 0 such that some n-ary operation w in Ω is a commutative n-group operation. Then the following statements are equivalent:*

(1) *$\langle A; \Omega, [\,]\rangle$ is cancellative.*

(2) *$\langle A^*; [\,]\rangle$ is a cancellative m-semigroup.*

(3) *$\langle A^*; [\,]\rangle$ is an m-subsemigroup.*

Definition 3.17. A commutative (Ω, m)-ringoid $\langle A; \Omega, [\,]\rangle$ is called an (Ω, m)-*field* if A^* is an m-group with respect to $[\,]$.

Theorem 3.18. *Let $\langle A; \Omega, [\,]\rangle$ be a generalized integral domain such that there exists an w in Ω such that $\langle A; w\rangle$ is a commutative n-group. Then $\langle A; \Omega, [\,]\rangle$ can be embedded into an (Ω, m)-field and the quotient ringoid A_{A^*} is (up to isomorphism) the unique minimal (Ω, m)-field with this property.*

We omit the above proofs for they are similar to the case of (n, m)-rings (see [**3**]).

We shall give an example which shows that in general the (Ω, m)-field need not be simple.

Example 3.20. Let $A = \langle\{1,2,3,4\}; w, \circ\rangle$ be the $(3,2)$-field where the multiplication table of $\circ$ is:

$\circ$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

and the ternary operation $w(x,y,z) = x \circ y \circ z$.

Then the congruence lattice $\mathrm{Con}(A)$ is isomporphic to the five-element modular lattice N_5. Thus A is not simple.

Hence, we propose the following:

Problem. Characterize those (Ω, m)-fields which are simple.

Leeson and Butson [**7**] gave a solution for the (Ω, m)-rings, but we do not know much about the general case.

Acknowledgement. The comments from the referee were helpful and greatly appreciated.

REFERENCES

[**1**] I. Adler, *Composition rings*, Duke Math. J. **29** (1962), 607–623.

[**2**] G. Crombez, *On (n,m)-rings*, Abh. Math. Sem. Univ. Hamburg, **37** (1972), 180–199.

[**3**] G. Crombez and J. Timm, *On (n,m)-quotient rings*, Abh. Math. Sem. Univ. Hamburg, **37** (1972), 200–203.

[**4**] H. Grell, *Beziehungen zwischen Idealen verschiedener Ringe*, Math. Annalen **97** (1927), 490–523.

[**5**] Sin-Min Lee, *On a simple extension of n-groupoids*, Southeast Asian Bull. Math. **2** (1978), 117–119.

[**6**] J. D. Monk and F. M. Sioson, *On the general theory of m-groups*, Fund. Math. **72** (1971), 233–244.

[**7**] J. J. Leeson and A. T. Butson, *On the general theory of (m,n)-rings*, Algebra Universalis **11** (1980), 42–76.

[**8**] J. J. Leeson and A. T. Butson, *Equationally complete (m,n)-rings*, Algebra Universalis **11** (1980), 28–41.

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(Received 14 10 1988)
(Revised 10 03 1989)