PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 46 (60), 1989, 25–32

ON SUMS INVOLVING RECIPROCALS OF CERTAIN LARGE ADDITIVE FUNCTIONS (II)

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Abstract. Let $\beta(n) = \sum_{p|n} p$ and $B(n) = \sum_{p^\alpha \| n} \alpha p$. Let $p(n)$ denote the largest prime factor of an integer $n \geq 2$. In the present paper we sharpen the asymptotic formula for the sum $\sum\limits_{2\leq n\leq x} B(n)/\beta(n)$ and we derive an asymptotic formula for the sum $\sum\limits_{2\leq n\leq x} (B(n)-\beta(n))/p(n)$.

1. Introduction and statement of results

Let $\beta(n) = \sum_{p|n} p$ and $B(n) = \sum_{p^\alpha \| n} \alpha p$.

In [**2**] it was proved that

$$\sum_{2\leq n\leq x} B(n)/\beta(n) = x + O\big(x\exp(-c_1(\log x \log_2 x)^{1/2})\big), \quad c_1 > 0, \tag{1.1}$$

and

$$\sum_{2\leq n\leq x} \beta(n)/B(n) = x + O\big(x\exp(-c_2(\log x \log_2 x)^{1/2})\big), \quad c_2 > 0. \tag{1.2}$$

The above results were slightly sharpened in [**6**]. Let us define $p(n)$ as the largest prime factor of $n \geq 2$, and $p(1) = 1$. In [**3**] it was proved that

$$\sum_{n\leq x} 1/p(n) = x\delta(x)\Big(1 + O((\log_2 x/\log x)^{1/2})\Big),$$

where

$$\delta(x) = \int_2^\infty \rho\left(\frac{\log x}{\log t}\right) t^{-2}\, dt.$$

Here $\rho(u)$ is the so-called "Dickman function", which is the solution of the differential-difference equation $u\rho'(u) + \rho(u-1) = 0$, $(u > 1)$, with the initial

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AMS Subject Classification (1980): Primary 10 H 15, (10 H 25)

condition $\rho(u) = 1$, $(0 \le u \le 1)$, $\rho(u)$ continuous at $u = 1$. An approximation to $\rho(u)$ in terms of elementary functions is

$$\rho(u) = \exp\bigl\{-u\bigl(\log u + \log_2 u - 1 + \log_2 u/\log u + O(1/\log u)\bigr)\bigr\}, \tag{1.3}$$

where $\log_2 u = \log\log u$. The asymptotic formula (1.3) was established by Hua [**5**] and de Bruijn [**1**], independently.

In [**8**], we proved that

$$\sum_{2\le n\le x} 1/\beta(n) = \bigl(D + O(\log_3^2 x/\log_2 x)\bigr)\sum_{n\le x} 1/p(n), \tag{1.4}$$

where $1/2 < D < 1$ is an absolute constant.

One of the aimes of the present paper is to provide sharpenings of (1.1) and (1.2). The results are contained in the following theorem.

Theorem 1.

$$\sum_{2\le n\le x}\frac{B(n)}{\beta(n)} = x + \frac{1}{2}D\log x\left(1 + O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}, \tag{1.5}$$

and

$$\sum_{2\le n\le x}\frac{\beta(n)}{B(n)} = x - \frac{1}{2}D\log x\left(1 + O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}, \tag{1.6}$$

where D is the same as in (1.4).

Moreover, in [**9**] we proved that

$$\sum_{2\le n\le x}\frac{1}{p^r(n)} = x\exp\left\{-(2r\log x\log_2 x)^{1/2}\left(1+\frac{\sqrt{r}}{2\sqrt{2}}\,\frac{\log_3 x}{\log_2 x}+O\left(\frac{1}{\log_2 x}\right)\right)\right\}, \tag{1.7}$$

and

$$\sum_{2\le n\le x}\frac{B(n)-\beta(n)}{p(n)} = x\exp\left\{-(2r\log x\log_2 x)^{1/2}\left(1+\frac{\sqrt{r}}{2\sqrt{2}}\,\frac{\log_3 x}{\log_2 x}+O\left(\frac{1}{\log_2 x}\right)\right)\right\}, \tag{1.8}$$

where $r > 0$ is arbitrary but fixed.

Another aim of the present paper is to provide sharpenings of (1.8). The result is:

Theorem 2.

$$\sum_{2\le n\le x}\frac{B(n)-\beta(n)}{p(n)} = \frac{1}{2}\log x\left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}.$$

By Theorem 1 and 2, we have

$$\sum_{2\le n\le x} \frac{B(n)-\beta(n)}{\beta(n)} \sim D\sum_{n\le x} \frac{B(n)-\beta(n)}{p(n)}.$$

It seems interesting to compare the sums involving reciprocals of $\beta(n)$ with the sums involving reciprocals of $p(n)$ [**8**]:

$$\sum_{2\le n\le x} \frac{1}{\beta(n)} \sim D\sum_{n\le x} \frac{1}{p(n)}, \quad \sum_{2\le n\le x} \frac{\omega(n)}{\beta(n)} \sim D\sum_{n\le x} \frac{\omega(n)}{p(n)},$$

and

$$\sum_{2\le n\le x} \frac{\Omega(n)-\omega(n)}{\beta(n)} \sim D\sum_{n\le x} \frac{\Omega(n)-\omega(n)}{p(n)},$$

where $\Omega(n)$ and $\omega(n)$ denote respectively the number of prime factors of n counted with and without multiplicities.

2. The necessary lemmas

Lemma 1 [**7**]. *Let*

$$L_1 = \exp\left[\left(\frac{1}{2}\log x \log_2 x\right)^{1/2}\left(1-2\frac{\log_3 x}{\log_2 x}\right)\right], \qquad \textit{and}$$
$$L_2 = \exp\left[\left(\frac{1}{2}\log x \log_2 x\right)^{1/2}\left(1+2\frac{\log_3 x}{\log_2 x}\right)\right].$$

Then we have

$$\sum_{n\le x} \frac{1}{p(n)} = \sum_{L_1<p\le L_2} \frac{1}{p}\Psi\left(\frac{x}{p},p\right)\left(1+O(\log^{-A} x)\right),$$

where $A>0$ is arbitrary but fixed, and $\Psi(x,y)$ denotes the number of positive integers not exceeding x, all of whose prime factors do not exceed y.

Lemma 2 [**4**]. *For any fixed $\varepsilon>0$ and $x\ge 3$, $\exp\{(\log_2 x)^{5/3+\varepsilon}\}\le y\le x$, we have uniformly*

$$\Psi(x,y) = x\rho(u)\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right), \qquad u=\frac{\log x}{\log y}.$$

Lemma 3 [**8**]. *For any fixed $\varepsilon>0$ and $1\le d\le y$, $\exp\{(\log_2 x)^{5/3+\varepsilon}\}\le y\le x^{1/2}$ we have uniformly*

$$\Psi(x/d,y) = \Psi(x,y)d^{-\beta}\left(1+O(1/u)+\left(\frac{\log(u+1)}{\log y}\right)\right), \tag{2.1}$$

where

$$\beta = \beta(x, y) = 1 - \frac{\xi(\log x / \log y)}{\log y};$$

here $\xi(u)$ *denotes the positive solution of the equation*

$$e^{\xi} = u\xi + 1, \qquad (u > 1), \tag{2.2}$$

and satisfies

$$\xi(u) = \log u + O(\log_2(u+2)), \quad u \to \infty. \tag{2.3}$$

Lemma 4 [**8**]. *For any fixed* $\varepsilon > 0$ *and*

$$1 \le d \le y, \quad \exp\{(\log_2 x)^{5/3+\varepsilon}\} \le y \le x,$$

we have uniformly

$$\Psi(x/d, y) \ll \Psi(x, y) d^{-\beta}.$$

3. Proofs of the Theorems

We shall only give a detailed proof of Theorem 1, since Theorem 2 may be obtained in a similar and simpler way.

By the definition of $B(n)$ and $\beta(n)$ we have (p, q denote primes):

$$\begin{aligned} W(x) := \sum_{2 \le n \le x} \frac{B(n) - \beta(n)}{\beta(n)} &= \sum_{q^\alpha \le x} (\alpha - 1) q \sum_{2 \le n \le x,\, q^\alpha \| n} \frac{1}{\beta(n)} \\ &= \sum_{q^\alpha \le x} (\alpha - 1) q \sum_{q < p_1 \le x/q^\alpha} \sum_{m_1 \le x/q^\alpha p_1,\, p(m_1) \le p_1,\, (q, m_1) = 1} \frac{1}{q + \beta(m_1 p_1)} \\ &\quad + O\biggl(\sum_{q^\alpha \le x} (\alpha - 1) \Psi(x q^{-\alpha}, q) \biggr) = W_1 + O(W_2), \quad \text{say.} \end{aligned} \tag{3.1}$$

It is evident that

$$W_1 = \sum_{p_1 \le x} \sum_{\substack{q^\alpha \le x/p_1, \\ q < p_1}} (\alpha - 1) q \sum_{\substack{m_1 \le x/q^\alpha p_1 \\ p(m_1) \le p_1,\, (q, m_1) = 1}} \frac{1}{q + \beta(m_1 p_1)}.$$

We may write

$$W_1 = \sum_{p_1 \le z_1} + \sum_{z_1 < p_1 \le L_1} + \sum_{L_1 < p_1 \le L_2} + \sum_{L_2 < p_1 \le x} = W_3 + W_4 + W_5 + W_6, \tag{3.2}$$

where $z_1 = \exp\{(1/10)(\log x \log_2 x)^{1/2}\}$, and L_1 and L_2 are defined in Lemma 1. Let $R = (\log x \log_3 x / \log_2 x) \sum_{n \le x} 1/p(n)$; we have

$$W_3 \le \sum_{p_1 \le z_1} \frac{1}{p_1} \sum_{q^\alpha \le x/p_1,\, q < p_1} (\alpha - 1) q \Psi(x q^{-\alpha} p_1^{-1}, p_1)$$

$$\ll \log^2 x \Big(\sum_{p_1 \le z_1} \frac{1}{p_1}\Big) \sum_{q \le z_1} q\Psi(xq^{-2}, z_1) \tag{3.3}$$
$$\ll x \exp\{-4(\log x \log_2 x)^{1/2}\} \ll R,$$

since by Lemma 2 and (1.3) we have $\Psi(xq^{-2}, z_1) \ll xq^{-2}\exp\{-4.5(\log x \log_2 x)^{1/2}\}$. Using Lemma 4 we have

$$\begin{aligned}
W_4 &\le \sum_{z_1 < p_1 \le L_1} \frac{1}{p_1} \sum_{q^\alpha \le x/p_1,\, q<p_1} (\alpha-1) q \Psi(xq^{-\alpha}p_1^{-1}, p_1) \\
&\ll \sum_{z_1 < p_1 \le L_1} \frac{1}{p_1}\Psi(x/p_1, p_1) \sum_{q^\alpha \le x/p_1,\, q<p_1} (\alpha-1) q^{1-\alpha(1-\delta')} \\
&\ll \sum_{z_1 < p_1 \le L_1} \frac{1}{p_1}\Psi(x/p_1, p_1) \sum_{q<p_1} q^{-1+2\delta'},
\end{aligned}$$

where $\delta' = (\log p_1)^{-1}\xi\big(\log(x/p_1)/\log p_1\big)$. By (2.2) and (2.3) we have

$$q^{2\delta'} \le \exp\left(2\xi\left(\frac{\log(x/p_1)}{\log p_1}\right)\right) \ll \log x \log_2 x,$$

for $z_1 < p_1 \le L_1$. Therefore using Lemma 1 we obtain:

$$W_4 \ll \log^2 x \sum_{z_1 < p_1 \le L_1} \frac{1}{p_1}\Psi(x/p_1, p_1) \ll R. \tag{3.4}$$

Similarly we have $W_6 \ll R$.

Now we come to the estimation of W_5 in (3.2). We consider separately the cases $p(m_1) < p_1$ and $p(m_1) = p_1$ and obtain

$$\begin{aligned}
W_5 = &\sum_{L_1 < p_1 \le L_2} \sum_{q^\alpha \le x/p_1,\, q<p_1} (\alpha-1)q \\
&\times \sum_{m_1 \le x/q^\alpha p_1,\, p(m_1)<p_1,\, (q,m_1)=1} \frac{1}{q+p_1+\beta(m_1)} \\
&+ O\Big(\sum_{L_1 < p_1 \le L_2} \sum_{q^\alpha \le x/p_1,\, q<p_1} \frac{(\alpha-1)q}{p_1}\Psi(xq^{-\alpha}p_1^{-2}, p_1)\Big).
\end{aligned} \tag{3.5}$$

Denoting by W_5' the main term on the right-hand side of (3.5) we may write

$$W_5' = \sum_{L_1 < p(m_1) < p_1,\, p(m_1)||m_1} + \sum_{L_1 < p(m_1) < p_1,\, p^2(m_1)||m_1} + \sum_{p(m_1) \le L_1}.$$

Then we have

$$W_5' = \sum_{L_1 < p_1 \le L_2} \sum_{L_1 < p_2 < p_1} \sum_{\substack{q^\alpha \le x/p_1p_2 \\ q<p_1,\, q \ne p_2}} (\alpha-1)q \sum_{\substack{m_2 \le x/q^\alpha p_1 p_2 \\ p(m_2)<p_2,\, (q,m_2)=1}} \frac{1}{q+p_1+p_2+\beta(m_2)}$$

$$+ O\Bigg(\sum_{L_1<p_1\le L_2} \sum_{L_1<p_2<p_1} \sum_{q^\alpha \le x/p_1p_2,\, q<p_1} (\alpha-1)qp_1^{-1}\Psi(x/q^\alpha p_1p_2^2, p_2) \Bigg)$$
$$+ O\Bigg(\sum_{L_1<p_1\le L_2} \sum_{p_2\le L_1} \sum_{q^\alpha \le x/p_1p_2,\, q<p_1} (\alpha-1)qp_1^{-1}\Psi(x/q^\alpha p_1p_2, p_2) \Bigg).$$

Proceeding as before, we obtain

$$W_5 = W_5'' + O\Bigg(\sum_{j=1}^{s} W_{7j} \Bigg) + O\Bigg(\sum_{j=2}^{s} W_{8j} \Bigg) + O\Bigg(\sum_{j=1}^{s-1} W_{9j} \Bigg), \tag{3.6}$$

where
(3.7)
$$W_5'' = \sum_{p_1,\dots,p_s} \sum_{\substack{q^\alpha \le x/p_1\dots p_s \\ q<p_1}} (\alpha-1)q \sum_{\substack{m_s\le x/q^\alpha p_1\dots p_s \\ p(m_s)<p_s,\,(q,m_s)=1}} \frac{1}{q+p_1+\cdots+p_s+\beta(m_s)},$$

where the ranges of summation in the above sums $p_1,\dots,p_s$ are $L_1<p_1\le L_2$, $L_1<p_2<p_1$, $\dots$, $L_1<p_s<p_{s-1}$, and $s\le \log_3 x$ is a large number which will be chosen later and

$$W_{7j} = \sum_{p_1,\dots,p_j} \sum_{q^\alpha\le x/p_1\dots p_j,\, q<p_1} \frac{(\alpha-1)q}{p_1}\Psi(x/q^\alpha p_1\dots p_{j-1}p_j^2, p_j), \tag{3.8}$$

$$W_{8j} = \sum_{p_1,\dots,p_{j-1}} \sum_{p_j\le L_1} \sum_{q^\alpha\le x/p_1\dots p_j,\, q<p_1} \frac{(\alpha-1)q}{p_1}\Psi(x/q^\alpha p_1\dots p_j, p_j), \tag{3.9}$$

$$W_{9j} = \sum_{p_1,\dots,p_j} \sum_{q^\alpha\le x/p_1\dots p_j,\, q<p_1} \frac{(\alpha-1)q}{p_1}\Psi(x/q^{\alpha+1} p_1\dots p_j, q). \tag{3.10}$$

Since

$$\frac{1}{q+p_1+\cdots+p_s+\beta(m_s)} = \frac{1}{p_1+\cdots+p_s} + O(qp_1^{-2}) + O(p_1^{-2}\beta(m_s)),$$

and

$$\sum_{\substack{m_s\le x/q^\alpha p_1\dots p_s \\ p(m_s)<p_s,\,(q,m_s)=1}} 1 = \Psi\left(\frac{x}{q^\alpha p_1\dots p_s}, p_s\right) - \Psi\left(\frac{x}{q^{\alpha+1} p_1\dots p_s}, p_s\right) + O(W_{7s}),$$

we have further

$$W_5'' = \sum_{p_1,\dots,p_s} \sum_{\substack{q^\alpha\le x/p_1\dots p_s \\ q<p_1,\,\alpha\ge 2}} \frac{q}{p_1+\cdots+p_s}\Psi(x/q^\alpha p_1\dots p_s, p_s)$$

$$+ O\Bigg(\sum_{p_1,\dots,p_s} \sum_{q^\alpha \le x/p_1\dots p_s,\, q<p_1} \frac{(\alpha-1)q^2}{p_1^2} \Psi(x/q^\alpha p_1\dots p_s, p_s) \Bigg)$$
$$+ O\Bigg(\sum_{p_1,\dots,p_s} \sum_{q^\alpha \le x/p_1\dots p_s,\, q<p_1} \frac{(\alpha-1)q}{p_1^2} \sum_{\substack{m_s \le x/q^\alpha p_1\dots p_s \\ p(m_s)<p_s,\,(q,m_s)=1}} \beta(m_s) \Bigg)$$
$$+ O(W_{7s}) = W_{10} + O(W_{11}) + O(W_{12}) + O(W_{7s}), \quad \text{say.}$$

We estimate first W_{10}. We consider separately the cases $\alpha = 2$ and $\alpha \ge 3$ and by using Lemmas 3 and 4 we obtain

$$W_{10} = \sum_{p_1,\dots,p_s} \frac{\Psi(x/p_1\dots p_s, p_s)}{p_1+\cdots+p_s} \sum_{q<p_1} q^{-1+2\delta_1}\big(1+O(\log_3 x/\log_2 x)\big)$$
$$+ O\Bigg(\sum_{p_1,\dots,p_s} \frac{\Psi(x/p_1\dots p_s, p_s)}{p_1+\cdots+p_s} \sum_{\substack{q^\alpha \le x/p_1\dots p_s \\ q<p_1,\, \alpha\ge 3}} q^{1-\alpha(1-\delta_1)} \Bigg),$$

where

$$\delta_i = \frac{1}{\log p_s}\xi\bigg(\frac{\log(x/p_i\dots p_s)}{\log p_s}\bigg), \qquad i = 1,2,\dots,s.$$

Using partial summation and the prime number theorem we have

$$\sum_{q<p_1} q^{-1+2\delta_1} = \int_{e^{1/\delta_1}}^{p_1} z^{-1+2\delta_1}\log^{-1} z\, dz\big(1+O(\log_3 x/\log_2 x)\big)$$
$$= (\log_2 x)^{-1} p_1^{2\delta_1}\big(1+O(\log_3 x/\log_2 x)\big).$$

Similarly

$$\sum_{q^\alpha \le x/p_1\dots p_s,\, q<p_1,\, \alpha\ge 3} q^{1-\alpha(1-\delta_1)} \ll 1.$$

Therefore we obtain

$$W_{10} = \frac{1}{\log_2 x} \sum_{p_1,\dots,p_s} \frac{\Psi(x/p_1\dots p_s, p_s)}{p_1+\cdots+p_s} p_1^{2\delta_1}\big(1+O(\log_3 x/\log_2 x)\big).$$

By (4.13) of [**8**] we have $p^{\delta_i} = p^{\delta}(1+O(\log_3 x/\log_2 x))$ for $L_1 < p \le L_2$, where $\delta = \delta_s$. Moreover by (4.6), (4.16), (4.18) and (4.31) of [**8**], we have

$$\sum_{p_1,\dots,p_s} \frac{\Psi(x/p_1\dots p_s, p_s)}{p_1+\cdots+p_s} = \bigg(D + O\bigg(\frac{\log_3^2 x}{\log_2 x}\bigg)\bigg) \sum_{L_1<p\le L_2} \frac{\Psi(x/p,p)}{p}.$$

Similarly, in a way analogous to the above we have for W_{10}

$$W_{10} = D(\log_2 x)^{-1}\big(1+O(\log_3^2 x/\log_2 x)\big) \sum_{L_1<p\le L_2} p^{-1+2\delta}\Psi(xp^{-1},p)$$
$$= \frac{1}{2} D\log x\big(1+O(\log_3^2 x/\log_2 x)\big) \sum_{n\le x} \frac{1}{p(n)}. \tag{3.12}$$

Now we come to the estimation of W_{12} in (3.11). By the definition of $\beta(m)$ and Lemma 4 we have

$$
\begin{aligned}
W_{12} &\leq \sum_{p_1,\dots,p_s} \sum_{q^\alpha \leq x/p_1\dots p_s,\, q<p_1} \frac{(\alpha-1)q}{p_1^2} \sum_{p<p_s} p\Psi(x/q^\alpha p_1\dots p_s p, p_s) \\
&\ll \sum_{p_1,\dots,p_s} \sum_{p<p_s} \frac{p}{p_1^2} \Psi(x/p_1\dots p_s p, p_s) p_1^{2\delta_1} (\log_2 x)^{-1}.
\end{aligned}
$$

By (4.19) of [**8**]

$$
\sum_{p_1,\dots,p_s} \sum_{p<p_s} \frac{p}{p_1^2} \Psi(x/p_1\dots p_s p, p_s) \ll \frac{1}{\log_2 x} \sum_{L_1<p\leq L_2} \frac{\Psi(x/p,p)}{p} \ll R.
$$

Similarly

$$
W_{12} \ll R. \tag{3.13}
$$

Similarly we have also

$$
W_{11}, W_{7j}, W_{8j}, W_{9j} \ll R. \tag{3.14}
$$

By putting (3.12)–(3.14) into (3.11) and (3.11), (3.14) into (3.6) and finally (3.3), (3.4) and (3.6) into (3.2) we get

$$
W_1 = \frac{1}{2} D \log x \big(1 + O(\log_3^2 x/\log_2 x)\big) \sum_{n\leq x} \frac{1}{p(n)}.
$$

Moreover, it is easy to prove that

$$
W_2 \ll R,
$$

which completes the proof of Theorem 1.

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(Received 18 01 1989)