PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 46 (60), 1989, pp. 33–42

ON SOME INTEGRALS INVOLVING THE MEAN SQUARE FORMULA FOR THE RIEMANN ZETA-FUNCTION

Aleksandar Ivić

Abstract. Let $E(T)$ denote the error term in the mean square formula for the Riemann zeta-function $\zeta(s)$. Several mean value results involving $E(T)$ and $\zeta(1/2+iT)$ are obtained which elucidate the behaviour of these functions.

1. Introduction

For $T \ge 0$ let

$$E(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt - T\left(\log\frac{T}{2\pi} + 2\gamma - 1\right)$$

as usual denote the error term in the mean-square formula for the Riemann zeta-function $\zeta(s)$ on the critical line $\operatorname{Re} s = 1/2$ (γ is Euler's constant). Atkinson [**1**] discovered an explicit formula for $E(T)$ (see also Ch. 15 of [**7**]), which led to much subsequent research. Thus Heath-Brown [**5**] used Atkinson's formula to show that

$$\int_2^T E^2(t)\,dt = CT^{3/2} + R(T) \qquad \left(C = \frac{2\zeta^4(3/2)}{3\sqrt{2\pi}\zeta(3)}\right) \tag{1.1}$$

with $R(T) \ll T^{5/4}\log^2 T$. Recently Meurman [**9**] and Motohashi [**10**] independently improved Heath-Brown's result to

$$R(T) \ll T\log^5 T. \tag{1.2}$$

It is known that $E(T) \ll T^{\alpha+\varepsilon}$ for an exponent $\alpha < 1/3$ (Ch. 15 of [**7**] and any $\varepsilon > 0$. The best result hitherto seems to be $\alpha \le 139/429 = 0.3240093\ldots$, proved in [**3**] by applying two-dimensional exponential sum techniques of G. Kolesnik [**8**]. On the other hand, (1.1) gives at once $E(T) = \Omega(T^{1/4})$ (proved by another method by

AMS Subject Classification (1980): Primary 10 H 25

Good [**2**]), but much sharper results are obtained by Hafner-Ivić [**3**], [**4**]. Therein it is shown that

(1.3)
$$E(T) = \Omega_+\{(T\log T)^{1/4}(\log\log T)^{(3+\log 4)/4}\exp(-B\sqrt{\log\log\log T})\} \quad (B>0)$$

and

$$E(T) = \Omega_-\left\{T^{1/4}\exp\left(\frac{D(\log\log T)^{1/4}}{(\log\log\log T)^{3/4}}\right)\right\} \qquad (D>0), \tag{1.4}$$

and it may be reasonably conjectured that these bounds are close to the true maximal order of $|E(T)|$. In order to establish (1.3)–(1.4) Hafner-Ivić proved the asymptotic formula

$$\int_2^T E(t)\,dt = \pi T + G(T), \tag{1.5}$$

where

$$\begin{aligned} G(T) &= 2^{-3/2}\sum_{n\le N}(-1)^n d(n) n^{-1/2} \\ &\times\left(\operatorname{ar\,sinh}\sqrt{\frac{\pi n}{2T}}\right)^{-2}\left(\frac{T}{2\pi n}+\frac14\right)^{-1/4}\sin\bigl(f(T,n)\bigr) \\ &-2\sum_{n\le N'} d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-2}\sin\left(T\log\left(\frac{T}{2\pi n}\right)-T+\frac{\pi}{4}\right)+O(T^{1/4}) \end{aligned} \tag{1.6}$$

with $d(n)$ the number of divisors of n,

$$f(T,n) = 2T\operatorname{ar\,sinh}\sqrt{\frac{\pi n}{2T}} + (2\pi nT+\pi^2n^2)^{1/2} - \frac{\pi}{4},$$

$$\operatorname{ar\,sinh} x = \log\bigl(x+\sqrt{x^2-1}\bigr), \quad N' = N'(T,N) = \frac{T}{2\pi}+\frac{N}{2}-\left(\frac{N^2}{4}+\frac{NT}{2\pi}\right)^{1/2},$$

and $AT < N < A'T$ for any two fixed constants $0 < A < A'$. Actually this formula is the analogue of Atkinson's formula for $E(T)$ for the function $\int_2^T \dot{E}(t)\,dt$, but in deriving (1.3) and (1.4) the full force of (1.6) was not needed. Instead, it was sufficient to use

(1.7)
$$G(T) = 2^{1/4}\pi^{-3/4}T^{3/4}\sum_{n=1}^{\infty}(-1)^n d(n)n^{-5/4}\sin\left(\sqrt{8\pi nT}-\frac{\pi}{4}\right)+O(T^{2/3}\log T),$$

which follows on simplifying (1.6) by Taylor's formula. Since the series in (1.7) is absolutely convergent, it follows that $G(T) = O(T^{3/4})$. On the other hand, it was shown in [**3**] that $G(T) = \Omega_\pm(T^{3/4})$, so that the order of $G(T)$ is precisely determined. In addition, in [**3**] we proved the mean-square formula

$$\int_2^T G^2(t)\,dt = ET^{5/2} + O(T^{2+\varepsilon}) \quad \left(E = \frac{\zeta^4(5/2)}{5\pi\sqrt{2\pi}\zeta(5)}\right). \tag{1.8}$$

2. The basic method

The purpose of this paper is to evaluate certain integrals of the type

$$I = I(T,H) = \int_T^{T+H} f\bigl(E(t)\bigr)\Bigl|\zeta\Bigl(\frac{1}{2}+it\Bigr)\Bigr|^2\,dt, \tag{2.1}$$

where $T \ge T_0$, $0 \le H \le T$, and $f(t)$ is a given function which is continuous in $[T, T+H]$. The method for evaluating the integral I in (2.1) is very simple. Namely, if $F' = f$, then from the definition of $E(T)$ it follows that

$$\begin{aligned} I &= \int_T^{T+H} f\bigl(E(t)\bigr)\Bigl(E'(t) + \log\frac{t}{2\pi} + 2\gamma\Bigr)\,dt \\ &= F\bigl(E(T+H)\bigr) - F\bigl(E(T)\bigr) + \int_T^{T+H} f\bigl(E(t)\bigr)\Bigl(\log\frac{t}{2\pi} + 2\gamma\Bigr)\,dt. \end{aligned} \tag{2.2}$$

Therefore the problem is reduced to a simpler one, namely to the evaluation of the integral where $|\zeta|^2$ is replaced by $\log(t/2\pi) + 2\gamma$. If T and $T+H$ are points at which $E(T) = E(T+H)$, then (2.2) simplifies even further. As the first application we prove

Theorem 1.

$$\int_0^T E^2(t)\Bigl|\zeta\Bigl(\frac{1}{2}+it\Bigr)\Bigr|^2\,dt = C\Bigl(\log\frac{T}{2\pi} + 2\gamma - \frac{2}{3}\Bigr)T^{3/2} + O(T\log^6 T) \tag{2.3}$$

with C as in (1.1),

$$\int_0^T E^4(t)\Bigl|\zeta\Bigl(\frac{1}{2}+it\Bigr)\Bigr|^2\,dt \ll T^{2+\varepsilon}, \tag{2.4}$$

$$\int_0^T E^6(t)\Bigl|\zeta\Bigl(\frac{1}{2}+it\Bigr)\Bigr|^2\,dt \ll T^{5/2+\varepsilon}, \tag{2.5}$$

$$\int_0^T E^8(t)\Bigl|\zeta\Bigl(\frac{1}{2}+it\Bigr)\Bigr|^2\,dt \ll T^{3+\varepsilon}. \tag{2.6}$$

To prove (2.3) we apply (2.2) with $H = T$, $f(t) = t^2$, $F(t) = t^3/3$. Using the weak $E(t) \ll t^{1/3}$ and (1.1)–(1.2) it follows that

$$\begin{aligned} \int_T^{2T} E^2(t)\Bigl|\zeta\Bigl(\frac{1}{2}+it\Bigr)\Bigr|^2\,dt &= O(T) + \int_T^{2T} E^2(t)\Bigl(\log\frac{t}{2\pi} + 2\gamma\Bigr)\,dt \\ &= \Bigl(\int_0^t E^2(u)\,du\Bigr)\Bigl(\log\frac{t}{2\pi} + 2\gamma\Bigr)\Bigr|_T^{2T} - \int_T^{2T}\Bigl(\int_0^t E^2(u)\,du\Bigr)\frac{dt}{t} + O(T) \\ &= Ct^{3/2}\Bigl(\log\frac{t}{2\pi} + 2\gamma - \frac{2}{3}\Bigr)\Bigr|_T^{2T} + O(T\log^6 T). \end{aligned}$$

Replacing T by $T\cdot 2^{-j}$ and summing over $j = 1,2,\dots$ we obtain (2.3). The remaining estimates (2.4)–(2.6) are obtained analogously using

$$\int_0^T |E(t)|^A\,dt \ll T^{1+A/4+\varepsilon} \quad (0 \le A \le 35/4), \tag{2.7}$$

a proof of which is given in [**6**] and Ch. 15 of [**7**]. The upper bounds in (2.4)–(2.6) are close to best possible, since

$$\int_0^T |E(t)|^A \left|\zeta\left(\frac12+it\right)\right|^2 dt \gg \begin{cases} T^{1+A/4+\varepsilon} & (0 \le A < 2),\\ T^{1+A/4}(\log T) & (A \ge 2),\end{cases} \tag{2.8}$$

for any fixed $A \ge 0$ and any given $\varepsilon > 0$. For $A > 2$ this follows easily from (2.3) and Hölder's inequality for integrals. For $0 \le A \le 2$ we use again (2.3) and Hölder's inequality to obtain

$$\begin{aligned} T^{3/2}\log T &\ll \int_0^T E^{A/2}(t)\left|\zeta\left(\frac12+it\right)\right| E^{2-A/2}(t)\left|\zeta\left(\frac12+it\right)\right| dt\\ &\le \left(\int_0^T |E(t)|^A|\zeta(1/2+it)|^2\,dt\right)^{1/2}\\ &\quad\times\left(\int_0^T |E(t)|^{8-2A}\,dt\right)^{1/4}\left(\int_0^T |\zeta(1/2+it)|^4\,dt\right)^{1/4}. \end{aligned}$$

Using (2.7) (with A replaced by $8-2A$) and the classical bound $\int_0^T |\zeta\left(\frac12+it\right)|^4\,dt \ll T^{1+\varepsilon}$ we obtain the first part of (2.8).

3. The integral involving $E(t)$

Perhaps the most interesting application of our method is the evaluation of the integral

$$\int_0^T E(t)\left|\zeta\left(\frac12+it\right)\right|^2 dt. \tag{3.1}$$

The function $E(t)$ has the mean value π by (1.5) while $|\zeta(1/2+it)|^2$ has the average value $\log t$. Therefore the integral in (3.1) represents in some way the way that the oscillations of these very important functions superimpose. We shall prove the following

Theorem 2. *Let $U(T)$ be defined by*

$$\int_0^T E(t)\left|\zeta\left(\frac12+it\right)\right|^2 dt = \pi T\left(\log\frac{T}{2\pi}+2\gamma-1\right)+U(T), \tag{3.2}$$

and let $V(T)$ be defined by

$$\int_0^T U(t)\,dt = \frac{\zeta^4(3/2)}{3\sqrt{2\pi}\,\zeta(3)}T^{3/2}+V(T). \tag{3.3}$$

Then

$$U(T) = O(T^{3/4}\log T), \quad U(T) = \Omega_{\pm}(T^{3/4}\log T), \tag{3.4}$$

$$V(T) = O(T^{5/4}\log T), \tag{3.5}$$

and

$$\int_2^T U^2(t)\,dt = T^{5/2}P_2(\log T) + O(T^{9/4+\varepsilon}), \tag{3.6}$$

where $P_2(x)$ is a suitable quadratic function in x.

We begin the proof by noting that (2.2) gives

$$\int_2^T E(t)\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = \frac{1}{2}E^2(T) + \int_2^T E(t)\left(\log\frac{t}{2\pi}+2\gamma\right)\,dt + O(1).$$

Using (1.5) the integral on the right-hand side of this equality becomes

$$\int_2^T \pi\left(\log\frac{t}{2\pi}+2\gamma\right)\,dt + \int_2^T\left(\log\frac{t}{2\pi}+2\gamma\right)\,dG(t)$$

$$= \pi T\left(\log\frac{T}{2\pi}+2\gamma\right) - \pi\int_2^T dt + O(1) + G(T)\left(\log\frac{T}{2\pi}+2\gamma\right) - \int_2^T G(t)\frac{dt}{t},$$

hence

$$U(T) = \frac{1}{2}E^2(T) + G(T)\left(\log\frac{T}{2\pi}+2\gamma\right) - \int_2^T G(t)\frac{dt}{t} + O(1).$$

Using (1.6) and simple estimates for exponential integrals (Lemma (2.1) of [**7**]) it follows that

$$\int_2^T G(t)\frac{dt}{t} = O(T^{1/4}). \tag{3.7}$$

This gives at once

$$U(T) = \frac{1}{2}E^2(T) + G(T)\left(\log\frac{T}{2\pi}+2\gamma\right) + O(T^{1/4}). \tag{3.8}$$

Since $E(T) \ll T^{1/3}$, then using (1.7) and $G(T) = \Omega_{\pm}(T^{3/4})$ we obtain (3.4). Therefore the order of magnitude of $U(T)$ is precisely determined, and we pass on to the proof of (3.5) and (3.6). From (3.8) we have

$$\int_{T/2}^T U^2(t)\,dt = \int_{T/2}^T G^2(t)\left(\log\frac{t}{2\pi}+2\gamma\right)^2 dt + O\left(\int_{T/2}^T \left(E^4(t)+T^{1/2}\right)dt\right)$$
$$+ O\left(\int_{T/2}^T |G(t)|\left(E^2(t)+T^{1/4}\right)\log T\,dt\right).$$

Using (2.7) with $A = 4$ it is seen that the contribution of the first O-term above is $\ll T^{2+\varepsilon}$. To estimate the second O-term we use (1.8) and the Cauchy-Schwarz inequality. We obtain a contribution which is

$$\ll \log T \left(\int_{T/2}^{T} G^2(t)\,dt \left(\int_{T/2}^{T} E^4(t)\,dt + T^{3/2} \right) \right)^{1/2} \ll T^{9/4+\varepsilon}.$$

Integration by parts and (1.8) yield

$$\int_{T/2}^{T} G^2(t) \left(\log \frac{t}{2\pi} + 2\gamma \right)^2 dt = \left(Et^{5/2} + O(t^{2+\varepsilon}) \right) \left(\log \frac{t}{2\pi} + 2\gamma \right)^2 \Big|_{T/2}^{T}$$
$$-2 \int_{T/2}^{T} \left(Et^{3/2} + O(t^{1+\varepsilon}) \right) \left(\log \frac{t}{2\pi} + 2\gamma \right) dt = t^{5/2} P_2(\log t) \Big|_{T/2}^{T} + O(T^{2+\varepsilon}),$$

where $P_2(x) = a_0 x^2 + a_1 x + a_2$ with a_0, a_1, a_2 effectively computable. This means that we have shown that

$$\int_{T/2}^{T} U^2(t)\,dt = t^{5/2} P_2(\log t) \Big|_{T/2}^{T} + O(T^{9/4+\varepsilon}),$$

so that replacing T by $T2^{-j}$ and summing over $j = 0, 1, 2, \ldots$ we obtain (3.6). Probably the error term in (3.6) could be improved to $O(T^{2+\varepsilon})$ (in analogy with (1.8)), but this appears to be difficult.

It remains to estabilish (3.3) with the O-result (3.5). Integrating (3.8) with the aid of (1.1)–(1.2) we have

$$(3.9)\quad \int_{T/2}^{T} U(t)\,dt = \frac{1}{2} C \left(T^{3/2} - (T/2)^{3/2} \right) + O(T^{5/4}) + \int_{T/2}^{T} G(t) \left(\log \frac{t}{2\pi} + 2\gamma \right) dt.$$

Using (1.6) (with $N = T$) it is seen that the contribution of the sum $\sum_{n \le N'}$ to the last integrals is $O(T)$, while the contribution of the error term $O(T^{1/4})$ is trivially $O(T^{5/4} \log T)$. The contribution of the sum $\sum_{n \le N}$ is, after simplification by Taylor's formula,

$$\int_{T/2}^{T} 2^{-1/4} \pi^{-3/4} t^{3/4} \sum_{n=1}^{\infty} (-1)^n d(n) n^{-5/4} \sin \left(\sqrt{8\pi n t} - \frac{\pi}{4} \right) \left(\log \frac{t}{2\pi} + 2\gamma \right) dt$$
$$+ O(T^{5/4}) = 2^{-3/4} \pi^{-5/4} \sum_{n=1}^{\infty} (-1)^{n-1} d(n) n^{-7/4}$$
$$\times \int_{T/2}^{T} t^{5/4} \left(\log \frac{t}{2\pi} + 2\gamma \right) \frac{d}{dt} \left\{ \cos \left(\sqrt{8\pi n t} - \frac{\pi}{4} \right) \right\} + O(T^{5/4})$$
$$= 2^{-3/4} \pi^{-5/4} \sum_{n=1}^{\infty} (-1)^{n-1} d(n) n^{-7/4} \left\{ t^{5/4} \left(\log \frac{t}{2\pi} + 2\gamma \right) \cos \left(\sqrt{8\pi n t} - \frac{\pi}{4} \right) \right\} \Big|_{T/2}^{T}$$
$$-2^{-3/4} \pi^{-5/4} \sum_{n=1}^{\infty} (-1)^{n-1} d(n) n^{-7/4}$$

$$\times \int_{T/2}^{T} \left(\frac{5}{4} t^{1/4}\left(\log\frac{t}{2\pi}+2\gamma\right)+t^{1/4}\right)\cos\left(\sqrt{8\pi n t}-\frac{\pi}{4}\right)\,dt + O(T^{5/4})$$

$$= 2^{-3/4}\pi^{-5/4}\sum_{n=1}^{\infty}(-1)^{n-1}d(n)n^{-7/4}$$

$$\times\left\{t^{5/4}\left(\log\frac{t}{2\pi}+2\gamma\right)\cos\left(\sqrt{8\pi n t}-\frac{\pi}{4}\right)\right\}\Bigg|_{T/2}^{T} + O(T^{5/4})$$

if we use Lemma (2.1) of [**7**] to estimate the last integral above. Inserting this expression in (3.9) we obtain (3.3) with

$$\begin{aligned} V(T) &= 2^{-3/4}\pi^{-5/4}T^{5/4}\left(\log\frac{T}{2\pi}+2\gamma\right) \\ &\times\sum_{n=1}^{\infty}(-1)^{n-1}d(n)n^{-7/4}\cos\left(\sqrt{8\pi n t}-\frac{\pi}{4}\right)+W(T), \end{aligned} \tag{3.10}$$

where

$$W(T) = O(T^{5/4}\log T). \tag{3.11}$$

Noting that the series in (3.10) is absolutely convergent, we obtain at once (3.5). We remark that, in analogy with (3.4), $V(T) = \Omega_{\pm}(T^{5/4}\log T)$ should also hold.

Since the series in (3.10) may be shown to be $\Omega_{\pm}(1)$ by the method of [**3**] which gives $G(T) = \Omega_{\pm}(T^{3/4})$, the omega result $V(T) = \Omega_{\pm}(T^{5/4}\log T)$ will follow if (3.11) can be replaced by $W(T) = o(T^{5/4}\log T)$ as $T\to\infty$. To see that the last assertion holds we recall that the error term in (1.6) is $O(T^{1/4})$. But analyzing carefully the arguments of [**3**] it is found that the function standing for this error term is an oscillating one. Hence integration will show that the resulting expression is $o(T^{5/4})$ and thus its contribution to (3.9) will be $o(T^{5/4}\log T)$ as $T\to\infty$. The details of this analysis are fairly complicated and we omit them.

4. Some lower bounds

We proceed now to derive some lower bound results for the integrals considered in previous sections. The basis for this analysis is the fact that, crudely speaking, the function $E(t)$ may increase rapidly, but it can decrease only fairly slowly. This may be put into quantitative form by the following elementary argument. Let $L(t) = t\big(\log(t/(2\pi))+2\gamma-1\big)$. From the definition of $E(T)$ we have, for $0\le x\le T$ and $T\ge T_0$,

$$0 \le \int_T^{T+x}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = L(T+x)-L(T)+E(T+x)-E(T)$$

$$= \int_T^{T+x}\left(\log\frac{t}{2\pi}+2\gamma\right)dt + E(T+x)-E(T) \le x\log T + E(T+x)-E(T).$$

This gives

$$E(T) \le E(T+x) + x\log T \qquad (0 \le x \le T) \tag{4.1}$$

and considering $\int_{T-x}^{T}\left|\zeta\left(\frac12+it\right)\right|^2 dt$ we obtain analogously

$$E(T) \ge E(T-x) - x\log T \qquad (0 \le x \le T). \tag{4.2}$$

Integrating these inequalities over x for $0 \le x \le H,\ 0 \le H \le T$, we obtain

$$HE(T) \le \int_T^{T+H} E(t)\,dt + \frac12 H^2\log T \qquad (0 < H \le T) \tag{4.3}$$

and

$$HE(T) \ge \int_{T-H}^{T} E(t)\,dt - \frac12 H^2\log T \qquad (0 < H \le T). \tag{4.4}$$

Take now for T the points where (1.3) is attained. Then (4.3) yields

$$\int_T^{T+H} E(t)\,dt \gg H(T\log T)^{1/4}(\log\log T)^{(3+\log 4)/4}\exp\bigl(-B\sqrt{\log\log\log T}\bigr) \tag{4.5}$$

for

$$0 < H < C_1T^{1/4}(\log T)^{-3/4}(\log\log T)^{(3+\log 4)/4}\exp\bigl(-B\sqrt{\log\log\log T}\bigr) \tag{4.6}$$

with some suitable $C_1 > 0$ and some $T = T_n$ such that $\lim_{n\to\infty} T_n = \infty$.

From (4.3)–(4.4) and the Cauchy-Schwarz inequality it follows that

$$E^2(T) \le 2H^{-1}\int_T^{T+H} E^2(t)\,dt + \frac12 H^2\log^2 T \qquad (E(T) > 0)$$

and

$$E^2(T) \le 2H^{-1}\int_{T-H}^{T} E^2(t)\,dt + \frac12 H^2\log^2 T \qquad \bigl(E(T) < 0\bigr).$$

Combining the preceding two inequalities we obtain for all $T \ge T_0$

$$E^2(T) \le 2H^{-1}\int_{T-H}^{T+H} E^2(t)\,dt + H^2\log^2 T \qquad (0 < H \le T). \tag{4.7}$$

Taking again $T = T_n$ the sequence for which $\lim_{n\to\infty} T_n = \infty$ and (1.3) holds, we have that

$$\int_{T-H}^{T+H} E^2(t)\,dt \gg H(T\log T)^{1/2}(\log\log T)^{(3+\log 4)/2}\exp\bigl(-2B\sqrt{\log\log\log T}\bigr), \tag{4.8}$$

provided that (4.6) holds. Since $C(T+H)^{3/2} - C(T-H)^{3/2}$ (the difference of the main terms in (1.1)) is asymptotic to $3CHT^{1/2}$, this means that there are intervals in which the integral in (4.8) is substantially larger than $C_2HT^{1/2}$, which is the bound that one expects from (1.1). Incidentally, (4.7) sets a limit to the upper bound for the error term $R(T)$ in (1.1). Namely we can prove that

$$R(T) = \Omega\left\{T^{3/4}(\log T)^{-1/4}(\log\log T)^{3(3+\log 4)/4}\exp\big(-B_1\sqrt{\log\log\log T}\big)\right\} \tag{4.9}$$

with a suitable B_1. To see this suppose that (4.9) is not true, and let $S(T,B_1)$, denote the function appearing on the right-hand side of (4.9). Choose T $(\to\infty)$ such that (1.3) holds and put $H = \big(S(T,B_1)\log^{-2}T\big)^{1/3}$. Then (1.1) and (4.7) give

$$\begin{aligned}
&(T\log T)^{1/2}(\log\log T)^{(3+\log 4)/2}\exp\big(-2B\sqrt{\log\log\log T}\big)\\
&\ll T^{1/2} + H^{-1}\big(R(T+H) - R(T-H)\big) + H^2\log^2 T\\
&\ll T^{1/2} + T^{1/2}(\log T)^{-1/6}(\log\log T)^{(3+\log 4)/2}\\
&\quad\times\exp\left(-\frac{2}{3}B_1\sqrt{\log\log\log T}\right)(\log T)^{2/3}.
\end{aligned}\tag{4.10}$$

This is a contradiction if $B_1 > 3B$ and hence (4.9) is proved. Thus (4.9) answers the question posed in Ch. 15.4 of [**7**], and a weaker result of the same type was announced in Ch. VII of [**11**]. However, the argument given in [**11**] is not quite correct (only $E(T) = \Omega(T^{1/4})$ is not sufficient in view of the term $T^{1/2}$ in the middle estimate in (4.10)), as was pointed out to me by D. R. Heath-Brown. Both Heath-Brown and T. Meurman independently informed me in correspondence how (4.9) is possible if (1.3) is known, and I warmly thank them for this. Since (4.9) does not seem to have appeared in print before, it seemed appropriate to discuss it here in detail, as the proof given in our text is an easy consequence of (1.3) and (4.7).

We shall finally consider the integral in (3.2) over short intervals. Here (3.4) provides the right order of magnitude for the function $U(T)$. To obtain the lower bound we seek, multiply (4.1) by $|\zeta(1/2+iT+ix)|^2$ and integrate over x for $0\le x\le H$, $T^\varepsilon \le H \le T$. This gives

$$E(T)\int_T^{T+H}\left|\zeta\left(\frac12+it\right)\right|^2dt \le \int_T^{T+H}\big(E(t)+H\log T\big)\left|\zeta\left(\frac12+it\right)\right|^2dt. \tag{4.11}$$

But it is well-known (see Ch. 9 of [**7**]) that

$$\int_T^{T+H}\left|\zeta\left(\frac12+it\right)\right|^2dt \gg_\varepsilon H\log T \qquad (T^\varepsilon\le H\le T),$$

so that for $E(T) > H\log T$ (4.11) reduces to

$$E(T) \ll \frac{1}{H\log T}\int_T^{T+H}E(t)\left|\zeta\left(\frac12+it\right)\right|^2dt + H\log T. \tag{4.12}$$

Take again T $(\to\infty)$ for which (1.3) holds. Then (4.12) implies

$$\begin{aligned}
&\int_T^{T+H}E(t)\left|\zeta\left(\frac12+it\right)\right|^2dt\\
&\gg HT^{1/4}(\log T)^{5/4}(\log\log T)^{(3+\log 4)/4}\exp\big(-B\sqrt{\log\log\log T}\big)
\end{aligned}\tag{4.13}$$

for $0 < \varepsilon < 1/4$, where for some suitable $C_2 > 0$

$$T^{\varepsilon} \ll H \le C_2 T^{1/4} (\log T)^{-3/4} (\log\log T)^{(3+\log 4)/4} \exp\big(-B\sqrt{\log\log\log T}\big).$$

It may be remarked that results analogous to (4.9) and (4.13) can be obtained by the same method for the integral in (2.3).

REFERENCES

[**1**] F. V. Atkinson, *The mean value of the Riemann zeta-function*, Acta Math. **81** (1949), 353–376.

[**2**] A. Good, *Ein Ω-Resultat für das quadratische Mittel der Riemannschen Zetafunction auf der kritischen Linie*, Invent. Math. **41** (1977), 233–251.

[**3**] J. L. Hafner and A. Ivić, *On the mean-square of the Riemann zeta-function on the critical line*, J. Number Theory, **32** (1989), 151–191.

[**4**] J. L. Hafner and A. Ivić, *On some mean value results for the Riemann zeta-function*, Proc. International Number Theory Conf., Québec 1987, Walter de Gruyter, 1989, 348–358.

[**5**] D. R. Heath-Brown, *The mean value theorem for the Riemann zeta-function*, Mathematika **25** (1978), 177–184.

[**6**] A. Ivić, *Large values of the error term in the divisor problem*, Invent. Math. **71** (1983), 513–520.

[**7**] A. Ivić, *The Riemann zeta-function*, John Wiley and Sons, New York, 1985.

[**8**] G. Kolesnik, *On the method of exponent pairs*, Acta Arith. **45** (1985), 115–143.

[**9**] T. Meurman, *On the mean square of the Riemann zeta-function*, Quart. J. Math. (Oxford) (2) **38** (1987), 337–343.

[**10**] Y. Motohashi, *A note on the mean value of the zeta and L-functions IV*, Proc. Japan Acad. Ser. A. Math. Sci. **62** (1986), 311–313.

[**11**] E. C. Titchmarsh, *The theory of the Riemann zeta-function* (2nd. ed. revised by D. R. Heath-Brown), Clarendon Press, Oxford, 1986.

Katedra Matematike RGF-a Univerziteta u Beogradu
Djušina 7
11000 Beograd
Yugoslavia

(Received 11 10 1988)