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A PROPERTY OF GENERALIZED RAMANUJAN'S SUMS CONCERNING GENERALIZED COMPLETELY MULTIPLICATIVE FUNCTIONS

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Abstract. Let A be a regular convolution in the sense of Narkiewicz. A necessary and sufficient condition for a multiplicative function to be A-multiplicative (i.e. such that $f(n) = f(d)f(n/d)$ whenever $d \in A(n)$) is given in terms of generalized Ramanujan's sums. (With the Dirichlet convolution A-multiplicative functions are completely multiplicative.) In addition, another necessary and sufficient condition for a multiplicative function to be completely multiplicative is given in terms of generalized Ramanujan's sums as well. As an application a representation theorem in terms of Dirichlet series is given. The results of this paper generalize respective results of Ivić and Redmond.

Let A be a mapping from the set $\mathbf{N}$ of positive integers to the set of subsets of $\mathbf{N}$ such that for each $n \in \mathbf{N}$, $A(n)$ consists entirely of divisors of n. Then the A-convolution of arithmetical functions is defined by

$$(fAg)(n) = \sum_{d \in A(n)} f(d)g(n/d).$$

Narkiewicz [**3**] defined an A-convolution to be *regular* if

(a) the set of arithmetical functions is a commutative ring with unity with respect to the ordinary addition and the A-convolution,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the function 1, defined by $1(n) = 1$ for all n, has an inverse μ_A with respect to the A-convolution, and $\mu_A(n) = 0$ or -1 whenever n is a prime power.

By [**3**] it can be seen that an A-convolution is regular if and only if

(i) $A(mn) = \{de : d \in A(m),\ e \in A(n)\}$ for all $(m,n) = 1$,

(ii) for every prime power $p^\alpha > 1$ there is a divisor $t = \tau_A(p^\alpha)$ of a, called the *type* of p^α, such that

$$A(p^\alpha) = \{1, p^t, p^{2t}, \dots, p^{rt}\},$$

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where $rt = a$, and

$$A(p^{it}) = \{1, p^t, p^{2t}, \ldots, p^{it}\} \quad \text{for all } i = 0, 1, 2, \ldots, r-1.$$

The prime powers p^t are called *A-primitive prime powers.*

We assume throughout this paper that A is an arbitrary but fixed regular convolution. For example, the Dirichlet convolution D, where $D(n)$ is the set of all positive divisors of n, and the unitary convolution U, where $U(n) = \{d > 0 : d|n, (d, n/d) = 1\}$, are regular.

Yocom [**6**] defined an arithmetical function to be A-multiplicative if for each $n \in \mathbf{N}$

$$f(d)f(n/d) = f(n) \quad \text{for all } d \in A(n).$$

For example, the D-multiplicative functions are the well-known completely or (totally) multiplicative functiona and the U-multiplicative functions are the well-known multiplicative functions. All A-multiplicative functions are multiplicative.

Yocom [**6**] proved that the following statements are equivalent:

(I) f is A-multiplicative,

(II) for each $n \in \mathbf{N}$, $f(n) = \prod f(p^t)^{a/t}$, where $n = \prod p^a$ is the canonical factorization of n and $t = \tau_A(p^a)$,

(III) $f(gAh) = fgAfh$ for all arithmetical functions g, h.

The inverse of an arithmetical function f with $f(1) \neq 0$ is defined by

$$fAf^{-1} = f^{-1}Af = e,$$

where $e(1) = 1$ and $e(n) = 0$ for $n \geq 2$. Yocom [**6**] proved that a multiplicative function f with $f(1) \neq 0$ is A-multiplicative if and only if

$$f^{-1} = \mu_A f.$$

The generalized Möbius function μ_A [**3**] is the multiplicative function such that for each prime power $p^a (\neq 1)$

$$\mu_A(p^a) = \begin{cases} -1 & \text{if } \tau_A(p^a) = a, \\ 0 & \text{otherwise.} \end{cases}$$

Ramanujan's sum $C(m; n)$ is defined by

$$C(m; n) = \sum_a \exp(2\pi i a m/n),$$

where a runs over a reduced residue system modulo n. A well-known evaluation of Ramanujan's sum is

$$C(m; n) = \sum_{d|(m,n)} d\mu(n/d),$$

where μ is the Möbius function. This formula suggests that we define (cf. [**1**]) a generalization of Ramanujan's sum as follows:

$$S_{A,k}^{f,g}(m_1, m_2, \dots, m_u; n) = \sum_{d^k \in A((m_1,\dots,m_u),n^k)_{A,k}} f(d)g(n/d),$$

where $(a,b)_{A,k}$ is the greatest k-th power divisor of a which belongs to $A(b)$. In other words,

$$S_{A,k}^{f,g}(m_1, m_2, \dots, m_u, ; n) = \sum_{\substack{d \in A_k(n) \\ d^k | m_1,\dots,m_u}} f(d)g(n/d),$$

where $A_k(n) = \{d > 0 : d^k \in A(n^k)\}$.

It is known [**5**] that the A_k-convolution is regular since the A-convolution is regular.

In [**2**] A. Ivić gave necessary and sufficient conditions for a multiplicative function to be completely multiplicative in terms of Ramanujan's sum. He also gave a Ramanujan-expansion of a generalization of von Mangoldt function. In [**4**] D. Redmond generalized the results of A. Ivić [**2**]. The purpose of this note is to generalize further these results. We assume throughout that f is multiplicative.

Theorem 1. (a) *If f is A_k-multiplicative, then for all positive integers n and non-negative integers $m_1, m_2, \dots, m_u$*

$$\sum_{d \in A_k(n)} f(d)f(n/d)S_{A,k}^{h,\mu_{A_k}}(m_1, m_2, \dots, m_u; d) = \begin{cases} f(n)h(n) & \text{if } n^k | m_1, m_2, \dots, m_u, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The converse holds if $h(p^{a-t}) \neq h(1)$, $t = \tau_{A_k}(p^a)$, for all prime powers p^a such that $a \neq t$.

(b) *If f is completely multiplicative and $f(1) \neq 0$, then for all positive integers n_1, m and non-negative integers $n_2, \dots, n_u$*

$$\sum_{d|n_1} f^{-1}(d)f(n_1/d)S_{A,k}^{h,g}(n_1/d, n_2, \dots, n_u; m) = \begin{cases} f(n_1)h(a)g(m/a) & \text{if } n_1 = a^k,\ a \in A_k(m), n_1 | n_2, \dots, n_u, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

f^{-1} being the Dirichlet inverse of f. The converse holds if $k = 1$, $A = D$, g is of the form μg and $g(p)h(p^{a-1}) \neq 0$ for all prime powers p^a.

Theorem 2. *Suppose $a, n_2, \dots, n_u$ are non-negative integers and k, m, u are positive integers. Define*

$$\varepsilon = 0 \text{ if } \sum_{\substack{d \in A_k(m) \\ d^k | n_2,\dots,n_u}} \frac{h(d)g(m/d)}{d^k} = 0, \quad \text{and } = 1 \text{ otherwise.}$$

Then

$$k^a \sum_{\substack{d\in A_k(m)\\ d^k|n_2,\dots,n_u}} \frac{g(m/d)h(d)\log^a d}{d^{k(1+\varepsilon)}} = \sum_{n=1}^{\infty} \frac{S_{A,k}^{h,g}(n,n_2,\dots,n_u;m)}{n^{1+\varepsilon}} \sum_{r=1}^{\infty} \frac{\mu(r)\log^a(nr)}{r^{1+\varepsilon}}.$$

D. Redmond [**4**] proved the teorems for $A = D$, $k = u = 1$, $g = \mu$. Further, we obtain the theorems of A. Ivić [**2**] if we assume in addition that $h(n) = n$ for all n. Ivić [**2**] and Redmond [**4**] presented formulas (1) and (2) in terms of Dirichlet series.

Proof of Theorem 1. Throughout the proof we shall use the notation:

$$\chi_A(m;d) = \begin{cases} 1 & \text{if } d \in A(m), \\ 0 & \text{otherwise.} \end{cases}$$

(a) Suppose f is A_k-multiplicative. Then, by (III), we have

$$\begin{aligned} \sum_{d\in A_k(n)} f(d)f(n/d)S_{A,k}^{h,\mu_{A_k}}(m_1,\dots,m_u;d) &= \big(fA_k f S_{A,k}^{h,\mu_{A_k}}(m_1,\dots,m_u;\cdot)\big)(n) \\ &= f(n)\big(1A_k S_{A,k}^{h,\mu_{A_k}}(m_1,\dots,m_u;\cdot)\big)(n) \\ &= f(n)\left(1A_k\left(\chi_D\big(m_1;(\cdot)^k\big)\dots\chi_D\big(m_u;(\cdot)^k\big)h\right)A_k\mu_{A_k}\right)(n) \\ &= f(n)\chi_D(m_1;n^k)\cdots\chi_D(m_u;n^k)h(n), \end{aligned}$$

which proves (1).

Conversely, suppose (1) holds with $h(p^{a-t}) \neq h(1)$, $t = \tau_{A_k}(p^a)$, for all prime powers p^a such that $a \neq t$. We proceed by induction on a to prove that

$$f(p^{at}) = f(p^t)^a \tag{3}$$

for all A-primitive prime powers p^t and integers a with $\tau_{A_k}(p^{at}) = t$. It is clear that (3) holds for $a = 1$. Suppose (3) holds for $a < s$ and $\tau_{A_k}(p^{st}) = t$. (Note that then $\tau_{A_k}(p^{at}) = t$ for $a < s$.) Taking $n = p^{st}$, $m_1 = \cdots = m_u = p^{stk}$ in (1) gives

$$\begin{aligned} f(p^{st})h(1) + \sum_{i=1}^{s-1} f(p^{it})f(p^{(s-i)t})\big(h(p^{it}) - h(p^{(i-1)t})\big)& \\ + f(p^{st})\big(h(p^{st}) - h(p^{(s-1)t})\big) &= f(p^{st})h(p^{st}), \end{aligned}$$

which can be written as

$$\begin{aligned} &\sum_{i=1}^{s-2}\big(f(p^{it})f(p^{(s-i)t}) - f(p^{(i+1)t})f(p^{s-(i+1)t})\big)h(p^{it}) \\ &\quad + f(p^{(s-1)t})f(p^t)h(p^{(s-1)t}) - f(p^t)f(p^{(s-1)t})h(1) \\ &\quad + f(p^{st})h(1) - f(p^{st})h(p^{(s-1)t}) = 0. \end{aligned}$$

By the inductive assumption

$$f(p^{it})f(p^{(s-i)t}) - f(p^{(i+1)t})f(p^{s-(i+1)t}) = 0;$$

hence we have

$$\Big(f(p^t)f(p^{(s-1)t}) - f(p^{st})\Big)\Big(h(p^{(s-1)t}) - h(1)\Big) = 0,$$

which completes the induction. So (3) holds. Thus, by (II), f is A_k-multiplicative.

(b) Suppose f is completely multiplicative with $f(1) \neq 0$. Then, as $f^{-1} = \mu f$, we have

$$\begin{aligned}
&\sum_{d|n_1} f^{-1}(d)f(n_1/d)S^{h,g}_{A,k}(n_1/d, n_2, \dots, n_u; m)\\
&\quad = f(n_1)\sum_{cd=n_1}\mu(d)S^{h,g}_{A,k}(c, n_2, \dots, n_u; m)\\
&\quad = f(n_1)\sum_{cd=n_1}\mu(d)\sum_{a^k|c}\chi_{A_k}(m;a)\chi_D(n_2;a^k)\cdots\chi_D(n_u;a^k)h(a)g(m/a)\\
&\quad = f(n_1)\sum_{a^k bd=n_1}\mu(d)\chi_{A_k}(m;a)\chi_D(n_2;a^k)\cdots\chi_D(n_u;a^k)h(a)g(m/a)\\
&\quad = f(n_1)\sum_{a^k v=n_1}\chi_{A_k}(m;a)\chi_D(n_2;a^k)\cdots\chi_D(n_u;a^k)h(a)g(m/a)\sum_{bd=v}\mu(d)\\
&\quad = f(n_1)\sum_{a^k v=n_1}\chi_{A_k}(m;a)\chi_D(n_2;a^k)\cdots\chi_D(n_u,a^k)h(a)g(m/a)e(v).
\end{aligned}$$

This proves (2).

Conversely, suppose (2) holds when $k = 1$, $A = D$, g is of the form μg and $g(p)h(p^{a-1}) \neq 0$ for all prime powers p^a. Then we prove that

$$f(p^a) = f(p)^a \tag{4}$$

for all prime powers p^a. Clearly (4) holds for $a = 1$. Suppose $a > 1$. Taking $n_1 = \cdots = n_u = m = p^a$ in (2) gives

$$\sum_{i=0}^{a} f^{-1}(p^i)f(p^{a-i})\sum_{j=0}^{a-i} h(p^j)(\mu g)(p^{a-j}) = f(p^a)h(p^a)g(1),$$

that is,

$$f(p^a)\left(h(p^a)g(1) - h(p^{a-1})g(p)\right) - f^{-1}(p)f(p^{a-1})h(p^{a-1})g(p) = f(p^a)h(p^a)g(1).$$

Therefore

$$f(p^a) = f(p)f(p^{a-1}),$$

which proves (4). Thus f is completely multiplicative. Now the proof of Theorem 1 is complete.

Proof of Theorem 2. Writing (2) in terms of Dirichlet series gives

$$\sum_{r=1}^{\infty} f(r)\mu(r)r^{-s} \sum_{n=1}^{\infty} f(n)S_{A,k}^{h,g}(n, n_2, \dots, n_u; m)n^{-s} = \sum_{\substack{d\in A_k(m)\\ d^k|n_2,\dots,n_u}} f(d^k)h(d)g(m/d)d^{-ks}.$$

Take $f = 1$. Suppose $\mathrm{Re}(s)$ is large enough and differentiate the above equation a times with respect to s. Then we obtain

$$\sum_{i=0}^{a}\binom{a}{i}\sum_{r=1}^{\infty}(-1)^i\mu(r)r^{-s}\log^i r\sum_{n=1}^{\infty}(-1)^{a-i}S_{A,k}^{h,g}(n, n_2, \dots, n_u; m)n^{-s}\log^{a-i} n = (-k)^a \sum_{\substack{d\in A_k(m)\\ d^k|n_2,\dots,n_u}} h(d)g(m/d)d^{-ks}\log^a d. \tag{5}$$

As $s \to (1+\varepsilon)^+$

$$\sum_{r=1}^{\infty}\mu(r)r^{-s}\log^i r \to \sum_{r=1}^{\infty}\mu(r)r^{-(1+\varepsilon)}\log^i r.$$

Moreover

$$\begin{aligned}\sum_{n\le x} S_{A,k}^{h,g}(n, n_2, \dots, n_u; m) &= \sum_{n\le x}\sum_{\substack{d\in A_k(m)\\ d^k|n,n_2,\dots,n_u}} h(d)g(m/d)\\ &= \sum_{\substack{d\in A_k(m)\\ d^k|n_2,\dots,n_u}} h(d)g(m/d)\sum_{d^k l\le x} 1 = \sum_{\substack{d\in A_k(m)\\ d^k|n_2,\dots,n_u}} h(d)g(m/d)\left(\frac{x}{d^k}+O(1)\right)\\ &= x\sum_{\substack{d\in A_k(m)\\ d^k|n_2,\dots,n_u}} h(d)g(m/d)d^{-k} + O(1).\end{aligned}$$

Now, we are in the position to prove that

$$\sum_{n=1}^{\infty} S_{A,k}^{h,g}(n, n_2, \dots, n_u; m)n^{-(1+\varepsilon)}\log^{a-i} n \tag{6}$$

converges. By Abel's identity, we have

$$\begin{aligned}&\sum_{n\le x} S_{A,k}^{h,g}(n, n_2, \dots, n_u; m)n^{-(1+\varepsilon)}\log^{a-i} n\\ &= \Bigg(x\sum_{d\in A_k(m),\, d^k|n_2,\dots,n_u} h(d)g(m/d)d^{-k} + O(1)\Bigg)\frac{\log^{a-i} x}{x^{1+\varepsilon}}\end{aligned}$$

$$-\int_1^x \left(t \sum_{d \in A_k(m),\, d^k | n_2, \ldots, n_u} h(d) g(m/d) d^{-k} + O(1) \right) \frac{d}{dt} \left(\frac{\log^{a-i} t}{t^{1+\varepsilon}} \right) dt.$$

If

$$\sum_{\substack{d \in A_k(m) \\ d^k | n_2, \ldots, n_u}} h(d) g(m/d) d^{-k} = 0,$$

then $\varepsilon = 0$ and we obtain

$$\sum_{n \le x} S_{A,k}^{h,g}(n, n_2, \ldots, n_u; m) n^{-1} \log^{a-i} n$$
$$= \frac{\log^{a-i} x}{x} O(1) - \int_1^x \frac{d}{dt} \left(\frac{\log^{a-i} t}{t^{1+\varepsilon}} \right) O(1)\, dt = \frac{\log^{a-i} x}{x} O(1) \to 0 \quad \text{as } x \to \infty.$$

On the other hand, if

$$\sum_{\substack{d \in A_k(m) \\ d^k | n_2, \ldots, n_u}} h(d) g(m/d) d^{-k} = K \neq 0,$$

then $\varepsilon = 1$ and we obtain

$$\sum_{n \le x} S_{A,k}^{h,g}(n, n_2, \ldots, n_u; m)\, n^{-2} \log^{a-i} n$$
$$= \big(Kx + O(1)\big) \frac{\log^{a-i} x}{x^2} - K \int_1^x t \frac{d}{dt} \left(\frac{\log^{a-i} t}{t^2} \right) dt - \int_1^x \frac{d}{dt} \left(\frac{\log^{a-i} t}{t^2} \right) O(1)\, dt$$
$$= O\left(\frac{\log^{a-i} x}{x} \right) - K \int_1^x \frac{\log^{a-i} t}{t^2}\, dt,$$

which converges as $x \to \infty$. So we have proved that (6) converges.

Now, letting $s \to (1+\varepsilon)^+$ in (5) we get Theorem 2.

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