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STOCHASTIC STRUCTURE OF SOME COMPLETELY MONOTONE FUNCTIONS

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Abstract. We describe the stochastic structure of some completely monotone functions. The presented results are connected with stability in some rarefaction procedures [**3**].

Introduction. Let $\kappa(s) = s^{-1}\big(1 - \exp(-s)\big)$ $(\kappa(0) = 1)$ be the Laplace transform of $\mathcal{U}(0,1)$ measure, and

$$\lambda_1(s) = \exp\left\{-\int_0^s \kappa(u)\,du\right\}. \tag{1}$$

We will show that $\lambda_1(s)$ is the Laplace transform of the probability measure on $\mathbf{R}^+$, and give the precise construction of a random variable with such distribution.

THEOREM 1. *Let $\{X_n, n \geq 1\}$ be the markovian sequence of random variables given by*

$$X_1 : \mathcal{U}(0,1) \&\ X_{n+1}|X_n : \mathcal{U}(0, X_n) \quad (n \in \mathbf{N}).$$

Then

$$S = \sum_1^\infty X_n$$

exists in mean square and with probability one.

Proof. By induction on n we get that the density function for X_n is

$$f_n(x) = \big(\Gamma(n)\big)^{-1}\big(-\ln(x)\big)^{n-1}, \quad 0 < x < 1.$$

Therefore $\mathbf{E}X_n = 2^{-n}$, $\mathbf{E}X_n^2 = 3^{-n}$ for $n \geq 1$.

Let $S_n = X_1 + \cdots + X_n$. For $m > n$

$$\mathbf{E}|S_m - S_n|^2 = \sum_{n+1}^m \mathbf{E}(X_k^2) + \sum_{k \neq l} \mathbf{E}(X_k X_l).$$

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$\{S_n, n \geq 1\}$ is a Cauchy sequence in L_2 sense. Indeed, as
$$\mathbf{E}(X_k X_l) \leq \big(\mathbf{E}(X_k^2)\big)^{1/2}\big(\mathbf{E}(X_l^2)\big)^{1/2} = 3^{-(k+l)/2}$$
for every k and l, it follows that
$$\mathbf{E}|S_m - S_n|^2 \leq \left(\sum_{n+1}^{m} 3^{-k/2}\right)^2.$$
In this way, $\mathbf{E}|S_m - S_n|^2 \to 0$, $n, m \to \infty$, and we have proved that S exists in mean square.

S exists with probability one, too. That follows from the fact that
$$\sup_{m>n}\left(\sum_{n+1}^{m} X_k\right) \xrightarrow{P} 0, \quad n, m \to \infty.$$

Put $Y_n = X_{n+1}X_1^{-1}$, $n \geq 1$. In the following theorem we will prove that the sequence Y_n has the same stochastic structure as X_n.

THEOREM 2. *$\{Y_n,\ n \geq 1\}$ is a markovian sequence, independent of X_1, such that* $Y_1 : \mathcal{U}(0,1)\&\ Y_{n+1}|Y_n : \mathcal{U}(0, Y_n)$.

Proof. Let $y \in (0,1)$. Then
$$P\{Y_1 < y\} = \mathbf{E}\big(P\{X_2 < yX_1\}|X_1\big) = \mathbf{E}(y|X_1) = y$$
so Y_1: $\mathcal{U}(0,1)$ is independent of X_1. Furthermore,
$$P\{Y_{n+1} < y|Y_n\} = P\{X_{n+2} < yX_1|X_{n+1}X_1^{-1}\}.$$
As $\mathcal{F}(X_{n+1}X_1^{-1}) \subset \mathcal{F}(X_{n+1}, X_1)$, where $\mathcal{F}(\quad)$ denotes the σ-field generated by the random variable indicated between the brackets, we have
$$\begin{aligned}
P\{Y_{n+1} < y|Y_n\} &= \mathbf{E}\left(\mathbf{E}\big(I\{X_{n+2} < yX_1\}|X_{n+1}, X_1\big)|X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(\mathbf{E}\big(I\{X_{n+2} < yX_1\}|X_{n+1}\big)|X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(P\{X_{n+2} < yX_1|X_{n+1}\}|X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\big(yX_{n+1}^{-1}X_1|X_{n+1}^{-1}X_1^{-1}\big)\\
&= yY_n^{-1},
\end{aligned}$$
so $Y_{n+1}|Y_n : \mathcal{U}(0, Y_n)$. Let us prove that the sequence $\{Y_n\}$ is markovian:
$$P\{Y_{n+1} < y|Y_1, \dots, Y_n\} = P\{X_{n+2} < yX_1|X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}\}.$$
Since $\mathcal{F}(X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}) \subset \mathcal{F}(X_1, \dots, X_{n+1})$ the conditional distribution above is equal to
$$\begin{aligned}
&\mathbf{E}\left(\mathbf{E}\big(I\{X_{n+2} < yX_1\}|X_1, \dots, X_{n+1}\big)|X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(\mathbf{E}\big(I\{X_{n+2} < yX_1\}|X_{n+1}\big)|X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(yX_1X_{n+1}^{-1}|X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(yY_n^{-1}|Y_1, \dots, Y_n\right)\\
&= yY_n^{-1} = P\{Y_{n+1} < y|Y_n\},
\end{aligned}$$

so the statement is proved.

Let us consider the distribution of S. Having in mind the properties of the sequence $\{Y_n\}$, it is easy to prove

THEOREM 3. *The Laplace transform of the distribution of S is $\lambda_1(u)$, introduced in* (1). *Also,*

$$\lambda(u) = 1 - u\lambda_1(u) \quad \textit{and} \quad \lambda_1(u)\big(\lambda_1(qu)\big)^{-1}, \quad 0 < q < 1,$$

are Laplace transforms of some probability measures on $\mathbf{R}^+$.

Proof. As

$$\mathbf{E}\exp(-uS) = \mathbf{E}\mathbf{E}\big(\exp(-uS)|X_1\big) = \mathbf{E}\exp(-uX_1)\mathbf{E}\left(\exp\left(-uX_1\sum_{n=1}^{\infty} Y_n\right)\right)$$

we have $\theta(u) = \mathbf{E}\exp(-uS) = \mathbf{E}\exp(-uX_1)\theta(X_1 u)$.

Since $X_1 : \mathcal{U}(0,1)$ it follows that

$$\theta(u) = \int_0^1 \exp(-ux)\theta(ux)\,dx \qquad \text{or} \qquad u\theta(u) = \int_0^u \exp(-y)\theta(y)\,dy.$$

As

$$\mathbf{E}X_n = 2^{-n}, \quad \mathbf{E}S = \sum_1^{\infty} \mathbf{E}X_n < \infty$$

it follows that $\theta(u)$ is differentiable for all $u \geq 0$. Hence

$$\theta(u) + u\theta'(u) = \exp(-u)\theta(u).$$

The solution of that simple differential equation, with the initial condition $\theta(0) = 1$, is

$$\lambda_1(u) = \exp\left(-\int_0^u x^{-1}\big(1 - \exp(-x)\big)\,dx\right).$$

Now we prove that $\lambda(s) = 1 - s\lambda_1(s)$ is an L-transform of some probability measure on $\mathbf{R}^+$.

As $\lambda'(s) = -\exp(-s)\lambda_1(s)$ and $\exp(-s)$ is an L-transform of the distribution concentrated in the point $x = 1$, it follows that $-\lambda'(s) = \exp(-s)\lambda_1(s)$, is an L-transform of some probability measure on $\mathbf{R}^+$. In this way, $\big(-\lambda'(s)\big)$ is completely monotone (CM) and for all $n \geq 0$

$$(-1)^n\big(-\lambda'(s)\big)^{(n)} = (-1)^{n+1}\lambda^{n+1}(s) \geq 0$$

or

$$(-1)^n\lambda^{(n)}(s) \geq 0, \quad n \geq 1.$$

Let us show that $\lambda(s) \geq 0$. From 1.4.2. we have

$$s\lambda_1(s) = \int_0^s \exp(-y)\lambda_1(y)\,dy \leq \int_0^s \exp(-y)\,dy = 1 - \exp(-s) \leq 1,$$

which is equivalent to $\lambda(s) \geq 0$.

In this way, $\lambda(s)$ is a CM function with the property $\lambda(0) = 1$.

Finally we prove that $\lambda_1(s)\big(\lambda_1(qs)\big)^{-1}$ is an L-transform of some probability measure on $\mathbf{R}^+$ for every $q \in (0,1)$. Indeed

$$\begin{aligned}\lambda_1(s)\big(\lambda_1(qs)\big)^{-1} &= \exp\left\{-\int_0^s \kappa(u)\,du + \int_0^{qs} \kappa(u)\,du\right\}\\ &= \exp\left\{-\int_0^s \big(\kappa(u) - q\kappa(qu)\big)\,du\right\}\\ &= \exp\left\{-\int_0^\infty x^{-1}\big(1-\exp(-sx)\big)d\big(\min\{x,1\}-\min\{x,q\}\big)\right\}.\end{aligned}$$

It is obvious that $\min\{x,1\}-\min\{x,q\}$ is a measure on $\mathbf{R}^+$. It is so-called canonical measure of some infinitely divisible law [1].

Consider the distribution function with L-transform λ_1. It has been proved that $S \stackrel{\mathcal{D}}{=} X_1(1+S')$, where X_1 and S' are independent random variables, $S' \stackrel{\mathcal{D}}{=} S$ and $X_1 : \mathcal{U}(0,1)$. If $\mathbf{L}_1$ denotes the distribution function for S, then for $z > 0$

$$\mathbf{L}_1(z) = \iint_A dx\, d\mathbf{L}_1(y),$$

where $A = \{(x,y) \mid x(y+1) < z,\ 0 < x < 1,\ y > 0\}$. Therefore,

$$\mathbf{L}_1(z) = \int_0^{1\wedge z} \mathbf{L}_1(zx^{-1}-1)\,dx,$$

where $1 \wedge z = \min\{1,z\}$. For $0 < z \leq 1$

$$\mathbf{L}_1(z) = cz, \qquad c = \int_0^\infty \mathbf{L}_1(y)(1+y)^{-2}\,dy,$$

and for $z > 1$

$$\mathbf{L}_1(z) = z\int_{z-1}^\infty \mathbf{L}_1(y)(1+y)^{-2}\,dy.$$

If $\mathbf{l}_1$ denotes the density function of this probability law, it follows that $\mathbf{l}_1(z) = c$ for $0 < z \leq 1$ and

$$\mathbf{l}_1(z) = z^{-1}\big(\mathbf{L}_1(z) - \mathbf{L}_1(z-1)\big), \quad z > 1.$$

In this way, the distribution function $\mathbf{L}_1(z)$ can be determined by solving that differential equation over the intervals $(n, n+1]$, $n \in \mathbf{N}$.

Let $\mathbf{L}$ be the distribution function with the Laplace transform λ, introduced in Theorem 3. As $\mathbf{l}_1(z) = 1 - \mathbf{L}(z)$, it follows that $\mathbf{L}(z) = 1 - c$, $z \in (0,1]$. In this way, $\mathbf{L}(z)$ has the jump in zero, i.e. $\mathbf{L}(0+) - \mathbf{L}(0) = 1 - c$. Of course, $\mathbf{L}(z)$ is not the unique distribution on $\mathbf{R}^+$ with stationary distribution $\mathbf{L}_1(z)$ [2]. If we introduce

$\mathbf{F}(z) = c^{-1}\{\mathbf{L}(z) - (1-c)\}$ then $\mathbf{F}(z)$ is also a distribution function on $\mathbf{R}^+$ with the same stationary distribution $\mathbf{L}_1(z)$. At the same time, $\mathbf{F}$ is continuous in zero.

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