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A REMARK ON A CERTAIN CLASS OF ARITHMETIC FUNCTIONS

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Abstract. Let $a(n)$ be an arithmetic function such that

$$\sum_{n=1}^{\infty} a(n)/n^s = f(s)\log g(s) + h(s),$$

where $f(s)$ is analytic for $\mathrm{Re}(s) > 1/2$ and bounded for $\mathrm{Re}(s) \geq 1/2+\varepsilon$, $g(s)$ is a zeta-like function, $h(s)$ is analytic and bounded for $\mathrm{Re}(s) \geq 1/2+\varepsilon$. Then

$$\sum_{n\leq x} a(n) = x\left[b_1/\log x + \cdots + b_m/\log^m x + O(1/\log^{m+1} x)\right]$$

with arbitrary fixed $m \geq 1$, $b_1 = f(1)$ and computable constants $b_2, \cdots, b_m$.

In this note we consider the problem of asymptotic representation of the summatory function of functions from a certain class of arithmetic functions. The functions from the considered class have appeared in certain problems of number theory (see for example [**1**]), but it seems that the general formula for the summatory functions is not known. In this connection we would like to fill this gap. We give the analytic proof of our Theorem, although it is possible to give an elementary proof (see [**2**]).

Let $\mathcal{A}$ be a class of arithmetic functions $a(n)$ such that:

1. $a(n) = O\big((\log^k n)\big)$, where $k > 0$ is a fixed number;

2. $\sum_{n=1}^{\infty} a(n)/n^s = f(s)\log g(s) + h(s)$ for $\mathrm{Re}(s) > 1$, where $f(s)$ is analytic for $\mathrm{Re}(s) > 0$ and bounded for $\mathrm{Re}(s) \geq 1/2+\varepsilon$ with $\varepsilon > 0$, $g(s)$ is analytic for $\mathrm{Re}(s) > 0$ except $s = 1$, where it has a simple pole, $h(s)$ is analytic and bounded for $\mathrm{Re}(s) \geq 1/2+\varepsilon$,

3. There exists an absolute constant $a > 0$ and a positive function $L(T) = O\big((1/\log T)^a\big)$ such that $g(s)$ has no zeros whenever

$$1 - L(T) \leq \mathrm{Re}(s) \leq 2, \quad |\,\mathrm{Im}(s)| \leq T, \quad T \geq 2. \tag{1}$$

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Let us note that the functions $\beta_\alpha(n)/n^\alpha = \frac{1}{n^\alpha}\sum_{p|n} p^\alpha$ ($\alpha > 1/2$ and p is a prime number) belongs to the class $\mathcal{A}$:

$$\sum_{n=1}^{\infty} \frac{\beta_\alpha(n)/n^\alpha}{n^s} = \zeta(s+\alpha)\log\zeta(s) + \zeta(s+\alpha)G_\alpha(s)$$

(for details see [**3**]).

Theorem. *If $a(n)$ belongs to $\mathcal{A}$, then*

$$\sum_{n\le x} a(n) = x\left[b_1/\log x + \cdots + b_m/\log^m x + O(\log^{-m-1} x)\right] \tag{2}$$

with arbitrary fixed $m \ge 1$, $b_1 = f(1)$ and computable constants $b_2, \ldots, b_m$.

For the proof of Theorem we shall use the following lemma.

Lemma. *If the function $g(s)$ satisfies the conditions 2 and 3 from the definition of the class $\mathcal{A}$, then in the domain (1) with the disk $|s-1| \le (\log T)^{-1}$ removed*

$$\log g(s) = O\big((\log T)^c\big), \tag{3}$$

where c is an absolute constant.

This Lemma can be proved in a standard way with the use of the Borel-Caratheodory theorem (see [**4**]).

Proof of Theorem. Let us consider the contour L consisting of:

- L_1: the line segment $[1-\delta, 1-\delta+iT]$,
- L_2: the line segment $[1-\delta+iT, b+iT]$,
- L_3: the line segment $[b+iT, b-iT]$,
- L_4: the line segment $[b-iT, 1-\delta-iT]$,
- L_5: the line segment $[1-\delta-iT, 1-\delta]$,
- L_6, L_7: the lower and upper edges of the cut in the complex plane along the line $[1-\delta, 1-\rho]$,
- C_ρ: the positively oriented circle of the radius ρ centred at $s = 1$, with $s = 1-\rho$ removed.

Here, $\delta = \delta(T) = L(T)$ (see the definition of $\mathcal{A}$), $b = 1 + (1/\log x)$.

The function $(s-1)g(s)$ is analytic in the neighbourhood of $s = 1$, so on C_ρ we have $\log g(s) = O(|\log\rho|)$. Therefore, if $\rho \to 0$, then

$$\frac{1}{2\pi i}\int_{C_\rho} f(s)\log g(s)\frac{x^s}{s}\,ds \to 0. \tag{4}$$

By Theorem 1 on page 75 in [**5**] and Cauchy's residue theorem we get

$$\begin{aligned}\sum_{n\le x} a(n) &= \frac{1}{2\pi i}\int_{b-iT}^{b+iT}\big[f(s)\log g(s)+h(s)\big]x^s s^{-1}\,ds + R_1 + R_2, \qquad (5)\\ &= \frac{1}{2\pi i}\left[\int_{L_1}+\int_{L_2}+\int_{L_4}+\int_{L_5}+\int_{L_6}+\int_{L_7}\right] + R_1 + R_2,\end{aligned}$$

where $R_1 = O\big(x^b/T(b-1)\big)$, $R_2 = O\big(xA(x)\log x/T\big)$. It follows from Lemma that

$$\left(\int_{L_1}+\int_{L_5}\right) f(s)\log g(s)\frac{x^s}{s}\,ds = O\big(x^{1-\delta}(\log T)^{c+1}\big). \qquad (6)$$

Moreover

$$\left(\int_{L_2}+\int_{L_4}\right) f(s)\log g(s)\frac{x^s}{s}\,ds = O\left(\int_{1-\delta}^{1}\frac{x^\sigma}{T}(\log T)^c\,d\sigma\right) = O\big(x(\log T)^c/T\big), \qquad (7)$$

$$\int_{b-iT}^{b+iT} h(s)\frac{x^s}{s}\,ds = O(x^b/T). \qquad (8)$$

The main term of the asymptotic formula we get from the estimation of the integrals

$$\frac{1}{2\pi i}\left(\int_{L_6}+\int_{L_7}\right) f(s)\log g(s)\frac{x^s}{s}\,ds.$$

We have

$$\begin{aligned}\int_{L_6} f(s)\log g(s)\frac{x^s}{s}\,ds &= \int_{1-\delta}^{1}\Big[f(1)+f'(1)(s-1)+\cdots+\frac{f^{(m)}(1)}{m!}(s-1)^m\\ &\quad +O\big(|s-1|^{m+1}\big)\Big]\big[-\log(s-1)+R_1(m,s)\big]x^s\big[1+R_2(m,s)\big]\,ds\\ &= \int_{1-\delta}^{1}\Big[f(1)+f'(1)(s-1)+\cdots+f^{(m)}(1)(s-1)^m/m! \qquad (9)\\ &\quad +O\big(|s-1|^{m+1}\big)\Big]\big[-\log(1-s)+i\pi+R_1(m,s)\big]x^s\big[1+R_2(m,s)\big]\,ds,\end{aligned}$$

$$\begin{aligned}\int_{L_7} f(s)\log g(s)\frac{x^s}{s}\,ds &= \int_{1-\delta}^{1}[f(1)+f'(1)(s-1)+\cdots \qquad (10)\\ &\quad +f^{(m)}(1)(s-1)^m/m! + O\big(|s-1|^{m+1}\big)]\\ &\quad \times[\log(1-s)+i\pi+R_1(s,m)]x^s[1+R_2(s,m)]\,ds,\end{aligned}$$

where

$$R_1(s,m) = \sum_{j=1}^{m} c_j(s-1)^j + O\big(|s-1|^{m+1}\big),$$

$$R_2(s,m) = \sum_{j=1}^{m} d_j(s-1)^j + O\bigl(|s-1|^{m+1}\bigr).$$

By cancelling the terms with $\log(1-s)$ in the sums of (9) and (10) the proof reduces to the evaluation of the integrals

$$I_j = \int_{1-\delta}^{1} x^s(s-1)^j\, ds,\ j = 0, 1, \dots .$$

Now, obviously $I_0 = 1/\log x + O\bigl(1/\log^{m+1} x\bigr)$, and by an integration by parts we get

$$I_j = \frac{-j}{\log\, x} I_{j-1} + O\left(\frac{1}{\log^{m+1}} x\right).$$

Therefore

$$\frac{1}{2\pi i}\left(\int_{L_6} + \int_{L_7}\right) f(s)\log g(s)\frac{x^s}{s}\, ds =$$

$$= f(1)x/\log x + \cdots + b_m x/\log^m x + O(x/\log^{m+1} x). \tag{11}$$

The assertion of Theorem follows from $(5), (6), (7), (8), (11)$ with

$$T = \exp(\log^\gamma x)\ \ (\gamma > 0).$$

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