PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 46 (60), 1989, pp. 59–70

CLASSES OF WEIGHTED SYMMETRIC FUNCTIONS*

Tan Cao Tran

Abstract. We generalize the concept of the k-th symmetric difference in the sense of Stein and Zygmund to that of symmetric difference with respect to a weight system of order n and the concept of symmetrically continuous functions and symmetric functions to that of functions symmetric with respect to a weight system of order n. We also study the classes of even symmetry and odd symmetry consisting of functions whose limits to the right and to the left exist at each point; hence, their set of points of discontinuity is countable, and they are in Baire class one. The functions symmetric with respect to a fixed weight system W_n of order n form a linear space $V(W_n)$, and the subclass $B(W_n)$ consisting of bounded functions forms a Banach space with the norm $\|f\| = \sup |f(x)|$.

Classes of weighted symmetric functions

Definitions: We call a *weight system of order* n a set of real numbers

$$W_n = \{w_{-n}, \dots , w_{-1}, w_0, w_1, \dots , w_n\}$$

such that $\sum_{k=-n}^{n} w_k = 0$ and $|w_n| + |w_{-n}| > 0$.

We say that a weight system W_n is *even* if

$$w_{-k} = w_k, \quad k = 0, 1, \dots , n,$$

with $\sum_{k=1}^{n} w_k \neq 0$ and $w_0 = -2\sum_{k=1}^{n} w_k \neq 0$ and a weight system W_n is *odd* if

$$w_{-k} = -w_k, \quad k = 0, 1, \dots , n,$$

with $\sum_{k=1}^{n} w_k \neq 0$ and $w_0 = 0$.

We call *symmetric diference with respect to a weight system* W_n *of order* n *for a finite real function* $f(x)$ the following expression

$$\Delta f(x; W_n, h) = \sum_{k=-n}^{n} w_k f(x + kh/2).$$

AMS Subject Classification (1980): Primary 26 A 99

* This manuscript is a part of the author's doctoral dissertation which was written at the University of Missouri at Kansas City. Special thanks is extended to Professor James Foran for his suggestions, encouragement, and patience.

This symmetric difference includes the k-th symmetric difference introduced by Stein and Zygmund [**4**]:

$$\Delta k f(x,h) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f[x + (k-2i)h/2].$$

A function $f(x)$ will be called *symmetric wih respect to a weight system* W_n if

$$\lim_{h\to 0} \Delta f(x; W_n, h) = 0,$$

and then we write $f \in S(W_n)$.

We see clearly that if W_1 is odd, then $S(W_1)$ is the class of symmetrically continuous functions, that is, the class of functions satisfying the condition

$$\lim_{h\to 0} \big(f(x+h) - f(x-h) \big) = 0,$$

and if W_1 is even, $S(W_1)$ is the class of symmetric functions satisfying the condition

$$\lim_{h\to 0} \big(f(x+h) + f(x-h) - 2f(x) \big) = 0.$$

Remark: Some classes are equal.

Example 1: Consider

$$\begin{aligned} W_1 &= \{1, -2, 1\} \\ W_2 &= \{1, 0, -2, 0, 1\} \\ W_3 &= \{1, 0, 0, -2, 0, 0, 1\} \\ W_n &= \{1, 0, \dots, 0, -2, 0, \dots, 0, 1\}, \end{aligned}$$

then

$$S(W_1) = S(W_2) = S(W_3) = \cdots = S(W_n).$$

Example 2: Let

$$\begin{aligned} W_1 &= \{-1, 0, 1\} \\ W_2 &= \{-1, 0, 0, 0, 1\} \\ W_n &= \{-1, 0, \dots, 0, \dots, 0, 1\}; \end{aligned}$$

then

$$S(W_1) = S(W_2) = \cdots = S(W_n).$$

Example 3: Let $W_1 = \{1, -2, 1\}$ and let W_2 be an even weight system

$$W_2 = \{a, b, c, b, a\}$$

such that

$$a \neq 0, \; c = -2(a+b), \; |a| < |b| \quad \text{and} \quad w = b/a < 0.$$

Let $B(W_1)$ be the class of functions bounded and symmetric with respect to W_1; $B(W_2)$ the class of functions bounded and symmetric with respect to W_2. Then

$$B(W_2) = B(W_1).$$

Proof: Prop. 2, which follows this example, shows that

$$f \in B(W_1) \Rightarrow f \in B(W_2).$$

Now we prove

$$f \in B(W_2) \Rightarrow f \in B(W_1).$$

Since f is bounded, we may assume $f(0) = 0$. For if $f(0) \neq 0$, the function

$$g(x) = f(x) - f(0)$$

satisfies $g(0) = 0$ and

$$\begin{aligned} \lim_{h\to 0} \sum_{k=-2}^{2} w_{-k} g(kh/2) &= \lim_{h\to 0} \sum_{k=-2}^{2} w_k \left[f(kh/2) - f(0)\right] \\ &= \lim_{h\to 0} \sum_{k=-2}^{2} w_{-k} f(kh/2) - f(0) \sum_{k=-2}^{2} w_k = 0. \end{aligned}$$

There exists an $M > 0$ such that, for every x,

$$|f(x)| < M.$$

We show that if f is not symmetric with respect to W_1 at $x = 0$, then there is a contradiction.

Suppose there exists a sequence $h_n \to 0$ and

$$f(h_n) + f(-h_n) - 2f(0) > \varepsilon.$$

Since f is symmetric with respect to W_2, there exists a $\delta > 0$ such that for $|h| < \delta$

$$\left|a\big[f(-2h) + f(2h)\big] + b\big[f(-h) + f(h)\big]\right| < \varepsilon|a|.$$

Let $w = b/a$ with $|w| > 1$. Then

$$|f(-2h) + f(2h) + w\left[f(-h) + f(h)\right]| < \varepsilon.$$

Since $h_n \to 0$, we have, for n sufficiently large,

$$-\varepsilon < f(-2h_n) + f(2h_n) + w\left[f(-h) + f(h)\right] < \varepsilon. \tag{1}$$

But $-w > 1$, and thus we have

$$-w\varepsilon < -w\left[f(-h_n) + f(h_n)\right]. \tag{2}$$

From (1) and (2)

$$(-w - 1)\varepsilon < f(-2h_n) + f(2h_n). \tag{3}$$

We also have, for n sufficiently large,

$$-\varepsilon < f(-4h_n) + f(4h_n) + w\left[f(-2h_n) + F(2h_n)\right] < \varepsilon. \tag{4}$$

And by (3) it follows that

$$(w^2 + w)\varepsilon < -w\left[f(-2h_n) + f(2h_n)\right]. \tag{5}$$

From (4) and (5):

$$(w^2 + w - 1)\varepsilon < f(-4h_n) + f(4h_n).$$

Similarly, for n sufficiently large:

$$(-w^3 - w^2 + w - 1)\varepsilon < f(-8h_n) + f(8h_n).$$

In general, let

$$a_p = (-1)^p w^p + (-1)^p w^{p-1} + (-1)^{p-1} w^{p-2} + (-1)^{p-2} w^{p-3} + \cdots - 1.$$

Then for n sufficiently large it follows that

$$a_p\varepsilon < f(-2^p h_n) + f(2^p h_n).$$

We can choose p and n so that

$$a_p\varepsilon > M \quad \text{and} \quad 2^p|h_n| < \delta.$$

Then

$$f(2^p h_n) + f(2^p h_n) > M.$$

This is a contradiction.

Proposition 1. *If f is continuous, then $f \in S(W_n)$ for any weight system W_n.*

Proof: Since

$$\lim_{h\to 0} f(x + kh/2) = f(x), \quad k = -n, \ldots, n,$$

we have

$$\lim_{h\to 0} \sum_{k=-n}^{n} w_k f(x + kh/2) = f(x) \sum_{k=-n}^{n} w_k = 0.$$

Proposition 2. *If f is symmetric, then f is symmetric with respect to any even weight system of any order.*

Proof: Let W_n be any even weight system. Then

$$\begin{aligned}\Delta F(x; W_n, h) &= \sum_{k=-n}^{n} w_k f(x + kh/2) \\ &= \sum_{k=1}^{n} w_k \left[f(x + kh/2) + f(x - kh/2) - 2f(x)\right].\end{aligned}$$

Since

$$\lim_{h\to 0} f(x+h/2)+f(x-h/2)-2f(x)=0,$$

we have

$$\lim_{h\to 0} f(x+kh/2)+f(x-kh/2)-2f(x)=0, \quad k=0,1,\dots,n.$$

Thus

$$\lim_{h\to 0} \Delta f(x;W_n,h)=0.$$

Corollary. *$S_{\mathrm{sym}}=\bigcap_{n=1}^{\infty} S(\text{even-}W_n)$ where S_{sym} is the class of symmetric functions, and $S(\text{even-}W_n)$ is the class of functions symmetric with respect to an even system W_n.*

Proposition 3. *If f is symetrically continuous, then f is symmetric with respect to any odd weight system of any order.*

Proof: Let W_n be any odd weight system. Then

$$\Delta f(x;W_n,h)=\sum_{k=1}^{n} w_k\left[f(x+kh/2)-f(x-kh/2)\right].$$

Since

$$\lim_{h\to 0} f(x+h/2)-f(x-h/2)=0,$$

we have

$$\lim_{h\to 0} f(x+kh/2)-f(x-kh/2)=0.$$

Thus

$$\lim_{h\to 0} \Delta f(x;W_n,h)=0.$$

Corollary. *$S_{\mathrm{sc}}=\bigcap_{n=1}^{\infty} S(\text{odd-}W_n)$ where S_{sc} is the class of symmetrically continuous functions, and $S(\text{odd-}W_n)$ is the class of functions symmetric with respect to an odd system W_n.*

Proposition 4. *There are functions symmetric with respect to an even weight system of order 2 but not symmetric.*

Proof: 1) First we construct a bounded function, which is symmetric with respect to

$$W_2=\{1,1,-4,1,1\} \quad \text{at } x=0$$

but not symmetric at $x=0$.

Let c be a fixed positive number.

We define f as follows: $f(0)=0$. For $x\notin(-2c,2c)$, $f(x)=0$.

For $x>0$,

$$
\begin{array}{ll}
f(x) = 0 & \text{for } x = 2c, c, c/2, c/2^2, \dots, c/2^n \dots \\
f(x) = 1 & \text{for } x = 3c/2, 3c^3/2, \dots, 3c/2^{2n+1}, \dots \\
f(x) = -1 & \text{for } x = 3c/2^2, 3c/2^4, \dots, 3c/2^{2n} \dots .
\end{array}
$$

For $x < 0$:

$$
\begin{array}{lll}
f(x) = 0 & \text{for} & x = -2c, -c, -c/2, -c/2^2, \dots, -c/2^n, \dots \\
 & & x = -3c/2, -3c/2^2, 3c/2^3, \dots, 3c/2^n, \dots \\
f(x) = 1 & \text{for} & x = -7c/2^3, -7c/2^5, \dots, -7c/2^{2n+3}, \dots \\
 & & x = -5c/2^3, -5c/2^5, \dots, -5c/2^{2n+3}, \dots \\
f(x) = -1 & \text{for} & x = -7c/2^2, -7c/2^4, \dots, -7c/2^{2n}, \dots \\
 & & x = -5c/2^2, -5c/2^4, \dots, -5c/2^{2n}, \dots .
\end{array}
$$

Between any two consecutive points listed above, $f(x)$ is linear. Then we see easily that f is symmetric with respect to W_2 at $x = 0$ but not symmetric at $x = 0$.

2) We also construct an unbounded function, which is symmetric with respect to the system

$$W_2 = \{2, 1, -6, 1, 2\} \quad \text{at } x = 0$$

but not symmetric at $x = 0$.

Let $c > 0$ and let

$$
\begin{array}{ll}
f(x) = 1 & \text{for } x \in (-2c, 2c) \\
f(x) = 0 & \text{for } x = 0 \\
f(x) = 1 & \text{for } x = -2c, -c/2, -c/2^3, \dots, /2^{2n+1}, \dots \\
f(x) = 1/8 & \text{for } x = -c, -c/2^2, -c/2^4, \dots, -c/2^{2n}, \dots \\
f(x) = 1 & \text{for } x = 2c \\
f(x) = -33/8 & \text{for } x = c.
\end{array}
$$

At the points

$$x = c/2, c/2^2, c/2^3, \dots$$

we define $f(x)$ by the recursion formula

$$2[f(c/2^n) + f(-c/2^n)] + f(c/2^{n+1}) + f(-c/2^{n+1}) = 0$$

and we let $f(x)$ be linear between any two consecutive points listed above. This function is symmetric with respect to W_2 but not symmetric at $x = 0$. To show this, we need the lemma:

LEMMA. *For the function f given above and each $h = c/2^n$*

$$2[f(-2h) + f(2h)] + f(-h) + f(h) = 0 \tag{1}$$

$$2[f(-h) + f(h)] + f(-h/2) + f(h/2) = 0 \tag{2}$$

and, for any t in $[h/2, h]$, f satisifies

$$2[f(-2t) + f(2t)] + f(-t) + f(t) = 0.$$

Proof of the lemma: Let

$$f(-2h) = a_0,\ f(2h) = a'_0$$
$$f(-h) = a_1,\ f(h) = a'_1$$
$$f(-h/2) = a_2,\ f(h/2) = a'_2.$$

Thus

$$(1) \text{ holds} \Leftrightarrow\ 2(a_0 + {a'}_0) + a_1 + {a'}_1 = 0,$$
$$(2) \text{ holds} \Leftrightarrow\ 2(a_1 + {a'}_1) + a_2 + {a'}_2 = 0.$$

Let L_1 be the segment joining $(-2h, a_0)$ and $(-h, a_1)$; L_2 joining $(2h, {a'}_0)$ and $(h, {a'}_1)$; L_3 joining $(-h, a_1)$ and $(-h/2, a_2)$; L_3 joining $(h, {a'}_1)$ and $(h/2, a'_2)$.

$$\begin{aligned} L_1 &: \quad y - a_0 = (a_1 - a_0)(x + 2h)/h \\ L_2 &: \quad y - {a'}_0 = ({a'}_1 - {a'}_0)(x - 2h)/h \\ L_3 &: \quad y - a_1 = 2(a_2 - a_1)(x + h)/h \\ L_4 &: \quad y - {a'}_1 = -2({a'}_2 - {a'}_1)(x - h)/h. \end{aligned}$$

Thus,

$$\begin{aligned} f(-2t) &= (a_1 - a_0)(-2t + 2h) + a_0 \\ f(2t) &= -({a_1}' - {a'}_0)(2t - 2h) + {a'}_0 \\ f(-t) &= 2(a_2 - a_1)(-t + h)/h + a_1 \\ f(t) &= -2({a'}_2 - {a'}_1)(t - h)/h + {a'}_1. \end{aligned}$$

So for every t in $[h/2, h]$,

$$2[f(-2t) + f(2t)] + f(-t) + f(t) = 0.$$

This lemma applied to each step, shows that f is symmetric with respect to W_2 at $x = 0$. But by the recursion formula

$$f(c/2^{n+1}) + f(-c/2^{n+1}) = -2[f(c/2^n) + f(-c/2^n)],$$

we have:

$$\begin{aligned} f(c/2^{n+1}) + f(-c/2^{n+1}) &= (-2)[f(c) + f(-c)] \\ &= (-2)^{n+1}[f(2c) + f(-2c)] = 2(-2)^{n+1}. \end{aligned}$$

Thus f is not symmetric at $x = 0$.

Proposition 5. *There are functions symmetric with respect to an odd weight system of order 2 but not symmetrically continuous.*

Proof. 1) First we construct a bounded function symmetric with respect to

$$W_2' = \{-1, -1, 0, 1, 1\}.$$

Let $f(x)$ be the function of Part 1 in the proof of Prop. 4. Then we define

$$\begin{aligned} g(x) &= f(x) && \text{for } x \geq 0 \\ g(x) &= -f(x) && \text{for } x < 0. \end{aligned}$$

This function $g(x)$ is symmetric with respect to W_2' and not symetrically continuous at $x = 0$.

2) We also construct an unbounded function symmetric with respect to

$$W_2'' = \{-2, -1, 0, 1, 2\}.$$

Let $c > 0$ and let

$$\begin{aligned} f(x) &= -1 && \text{for } x \leq -2c \\ f(x) &= 0 && \text{for } x = 0 \\ f(x) &= -1 && \text{for } x = -2c, -c/2, -c/2^3, \dots, -c/2^{2n+1} \\ f(x) &= -1/8 && \text{for } x = -c, -c/2^2, -c/2^4, \dots, -c/2^{2n} \\ f(x) &= 1 && \text{for } x \geq 2c \\ f(x) &= -33/8 && \text{for } x = c. \end{aligned}$$

At the points

$$x = c/2,\ c/2^2,\ c/2^3,\ \dots,\ c/2^n,\ \dots$$

we define $f(x)$ by the recursion formula

$$2\big[f(c/2^n) - f(-c/2^n)\big] + f(c/2^{n+1}) - f(-c/2^{n+1}) = 0$$

and let $f(x)$ be linear between any two consecutive points listed above. Then by a proof similar to that of Part 2 in Prop. 4, we see that the function $f(x)$ defined here is symmetric with respect to the odd system W_2'' but not symmetrically continuous at $x = 0$.

Thus there is a variety of functions satisfying different weighted symmetric conditions.

PROPOSITION 6. *A function f is continuous iff f is both symmetric and symmetrically continuous.*

Proof: If f is continuous, by Prop. 1, f is both symmetric and symmetrically continuous.

Conversely, suppose f is both symmetric and symmetrically continuous, then

$$\lim_{h\to 0}\big(f(x + h/2) - f(x - h/2)\big) = 0 \tag{1}$$

$$\lim_{h\to 0}\big(f(x+h/2)+f(x-h/2)-2f(x)\big)=0. \tag{2}$$

From (1) and (2) we have

$$\lim_{h\to 0}\big(2f(x+h/2)-2f(x)\big)=0.$$

Thus, $f(x+0)=f(x-0)=f(x)$ and f is continuous.

PROPOSITION 7. *If f is symmetric with respect to an even (odd) weight system W_n of order n, and if at each point the limits to the right and to the left of f exist, then f is symmetric with respect to any even (odd) weight system of any order.*

Proof: Fix $n\geq 1$. Let f be symmetric with respect to an even or odd system W_n.

If $n=1$, then the results are true by Prop. 2 and Prop. 3. Suppose $n>1$ and W_n is even.

Since the right limits and left limits exist at each point x, we have

$$\lim_{h\to 0}\sum_{k=1}^{n} w_k\big[f(x+h/2)+f(x-h/2)-2f(x)\big]=0$$

$$\big[f(x+0)+f(x-0)-2f(x)\big]\sum_{k=1}^{n} w_k=0.$$

By the definition of an even weight system, we have

$$\sum_{k=1}^{n} w_k\neq 0.$$

Thus

$$f(x+0)+f(x-0)-2f(x)=0.$$

So f is symmetric, and by Prop. 2, f is symmetric with respect to any even system of any order.

If W_n is odd, we have

$$\lim_{h\to 0}\sum_{k=1}^{n} w_k\big[f(x+kh/2)+f(x-kh/2)\big]=0$$

and

$$\big[f(x+0)+f(x-0)\big]\sum_{k=1}^{n} w_k=0.$$

Since $\sum_{k=1}^{n} w_k\neq 0$, we have

$$f(x+0)-f(x-0)=0.$$

Thus f is symmetrically continuous and hence, by Prop. 3, f is symmetric with respect to any odd system of any order.

By Prop. 7, we see that functions which are symmetric with respect to an even (odd) weight system and whose limits to the right and to the left at each point exist form a special subclass of symmetric functions. There are only two such subclasses which we call even symmetry class and odd symmetry class denoted by S_{even} and S_{odd}.

1) *A function in the even* (*odd*) *symmetry class is symmetric with respect to any even* (*odd*) *weight system of any order.*

2) *If f is measurable and f is in S_{even} or S_{odd}, then the set of points of discontinuity of f is countable*; *hence, f is in Baire class one.* For it is well known that, for a function f, if the right limit and left limit exist at each point, then the set of points of discontinuity is countable, and f is, therefore, Baire 1 [**2**, p. 283].

3) $S_{\text{even}} \subset \bigcap_{n=1}^{\infty} S(\text{even-}W_n) = S_{\text{sym}}$.

The inclusion is strict; for instance, the function

$$\begin{aligned} f(x) &= 1 + \sin(1/x) && \text{for } x > 0 \\ f(x) &= 0 && \text{for } x = 0 \\ f(x) &= -1 + \sin(1/x) && \text{for } x < 0 \end{aligned}$$

is symmetric, but the right limit and left limit do not exist at $x = 0$.

4) $S_{\text{odd}} \subset \bigcap_{n=1}^{\infty} S(\text{odd-}W_n) = S_{\text{sc}}$.

The inclusion is strict because the function

$$\begin{aligned} f(x) &= \cos(1/x) && \text{for } x \neq 0 \\ f(x) &= 0 && \text{for } x = 0 \end{aligned}$$

is symmetrically continuous, but the right limit and left limit do not exist at $x = 0$.

Now consider the class of functions such that at each x,

$$\lim_{h \to 0+} \lim_{m \to 0+} \sum_{k=1}^{n} w_k \big[f(x + kh/2) + f(x - km/2) - 2f(x) \big] = 0$$

if W_n is even, and

$$\lim_{h \to 0+} \lim_{m \to 0+} \sum_{k=1}^{n} w_k \big[f(x + kh/2) + f(x - km/2) \big] = 0$$

if W_n is odd.

Then it is clear that *the left limit and the right limit of f exist at each point x. Hence, the set of points of discontinuity of f is countable, and f is Baire* 1.

Proposition 8. *If $\{f_i\}$ is a sequence of functions symmetric with respect to a fixed weight system W_n (W_n need not be even or odd), and if f is the uniform limit of $\{f_i\}$, then f is symmetric with respect to W_n.*

Proof. We have

$$\begin{aligned}\Delta f(x;W_n,h) &= \Delta f(x;W_n,h) - \Delta f_i(x;W_n,h) + \Delta f_i(x;W_n,h)\\ &= \sum_{k=-n}^{n} w_k\big[f(x+kh/2) - f_i(x+kh/2)\big] + \Delta f_i(x;W_n,h).\end{aligned}$$

Then

$$|\Delta f(x;W_n,h)| \le \sum_{k=-n}^{n} |w_k| \cdot |f(x+kh/2) - f_i(x+kh/2)| + |\Delta f_i(x;W_n,h)|.$$

Since f_i is symmetric with respect to W_n, there is a $\delta > 0$ such that

$$|\Delta f_i(x;W_n,h)| < \varepsilon/2 \quad \text{for } |h| < \delta.$$

Since $f_i \to f$ uniformly, there is an $N > 0$ such that

$$|f(x+kh/2) - f_i(x+kh/2)| < \varepsilon \cdot \left(2\sum_{k=-n}^{n}|w_k|\right)^{-1} \quad \text{for } i > N.$$

Thus

$$|\Delta f(x;W_n,h)| < \varepsilon \quad \text{for } |h| < \delta.$$

PROPOSITION 9. *Given a weight system W_n of order n, the sum of two functions symmetric with respect to W_n is symetric with respect to W_n.*

Proof: Suppose

$$\lim_{h\to 0} \Delta f(x;W_n,h) = 0, \quad \lim_{h\to 0} \Delta g(x;W_n,h) = 0;$$

then

$$\lim_{h\to 0} \Delta(f+g)(x;W_n,h) = \lim_{h\to 0} \Delta f(x;W_n,h) + \lim_{h\to 0} \Delta g(x;W_n,h) = 0.$$

PROPOSITION 10. *The product of a function symmetric with respect to a weight system W_n by a scalar is symmetric with respect to W_n.*

Proof: Suppose

$$\lim_{h\to 0} \Delta f(x;W_n,h) = 0.$$

Then

$$\lim_{h\to 0} \Delta(cf)(x;W_n,h) = c\lim_{h\to 0} \Delta f(x;W_n,h) = 0.$$

THEOREM. *Given a weight system W_n of order n, the functions symmetric with respect to W_n form a linear space $V(W_n)$. The uniform limit of a sequence in $V(W_n)$ belongs to $V(W_n)$. Moreover, if $B(W_n)$ is the subspace of $V(W_n)$ consisting of bounded functions, then $B(W_n)$ is a Banach space with the norm* $\|f\| = \sup|f(x)|$.

REFERENCES

[**1**] J. M. Ash, *Generalizations of Riemann derivative*, Trans. Amer. Math. Soc. **126** (1967), 181–199.

[**2**] F. Hausdorff, *Set Theory*, Chelsea, New York, 1959.

[**3**] L. Larson, *Symmetric Real analysis: A survey*, Real Anal. Exch. **9(1)** (1983/84), 154–178.

[**4**] E. M. Stein and A. Zygmund, *Smoothness and differentiability of functions*, Ann. Univ. Sci., Budapest, **III-IV** (1960–61), 295–307.

Department of Mathematics
Loras College
Dubuque, Iowa 52001, USA

(Received 14 07 1987)